Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 30, 2003*
Revised: *July 22, 2003*
Accepted: *July 29, 2003*

String field theory vertices for fermions of integral weight

Alexander Kling and Sebastian Uhlmann

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
E-mail: kling@itp.uni-hannover.de, uhlmann@itp.uni-hannover.de

Abstract: We construct Witten-type string field theory vertices for a fermionic first order system with conformal weights $(0, 1)$ in the operator formulation using delta-function overlap conditions as well as the Neumann function method. The identity, the reflector and the interaction vertex are treated in detail paying attention to the zero mode conditions and the U(1) charge anomaly. The Neumann coefficients for the interaction vertex are shown to be intimately connected with the coefficients for bosons allowing a simple proof that the reparametrization anomaly of the fermionic first order system cancels the contribution of two real bosons. This agrees with their contribution $c = -2$ to the central charge. The overlap equations for the interaction vertex are shown to hold. Our results have applications in N = 2 string field theory, Berkovits' hybrid formalism for superstring field theory, the $\eta\xi$-system and the twisted bc-system used in bosonic vacuum string field theory.

Keywords: D-branes, String Field Theory.

 http://jhep.sissa.it/archive/papers/jhep072003061/jhep072003061.pdf

Contents

1. Introduction

The study of nonperturbative physics in string theory has enjoyed a lively interest in recent years. String field theory, especially in the formulation of [1], has proven to be a useful tool for developing qualitative as well as quantitative results. In particular, Sen's conjectures on tachyon condensation [2]–[4] have set the stage for putting string field theory to work (see, for instance, the reviews [5]–[8] and references therein). In this framework it should be possible to describe the result of the condensation process, the closed string vacuum, as a solution to its equation of motion. However, no plausible candidate for a solution in open string field theory has been found yet.

In order to gain some insight into the structure of solutions describing D-branes, Rastelli, Sen and Zwiebach proposed to expand all string fields around a (unknown) closed string vacuum solution. This vacuum string field theory [9, 10] is described as a certain singular limit, in which the kinetic operator consists only of ghosts. In this limit, the equation of motion of bosonic string field theory factorizes into a matter and a ghost part, which can be solved independently; the matter part of the string field has to be a projector of the star algebra. This separation, however, does not pertain to open string field theory, where the kinetic operator mixes matter and ghost sectors. Much work has been done in the context of bosonic string theory to classify these solutions in terms of projectors of the star algebra [11]–[16], most prominently the sliver. They have been identified with (multiple) D-branes of various dimensions, and it was shown that they reproduce the correct ratio of

JHEP07(2003)061

tensions. Their similarity to noncommutative solitons has been investigated [17, 18]. In such considerations a major technical simplification is achieved by taking advantage of the map from Witten's star product to a continuous[1] Moyal product [21].

Much less is known for superstrings. Different generalizations of bosonic vacuum string field theory to non-BPS and D-$\bar{D}$ brane systems have been proposed [22]–[24] using Berkovits' formulation for a nonpolynomial superstring field theory [25] as well as cubic open superstring field theory [26, 27]. Although superstring field theories applied to the problem of tachyon condensation perform well [28]–[30], it is considerably harder to make sense of the conjectured versions around the tachyon vacuum. Despite considerable efforts [31]–[34] up to now no satisfactory solutions to vacuum superstring field theory have been given. Even worse, recent results using the numerical technique of level truncation seem to indicate that the pure ghost ansatz for the kinetic operator fails to describe the theory around the tachyon vacuum [35]. Therefore it is necessary to gain more insight into the structure of solutions to string field theory with more general kinetic operators mixing different sectors of the theory. This is, of course, a technically demanding task.

Due to these difficulties it seems worthwhile to take an apparent sidestep and to consider alternative approaches to tackle the problems stated above. In order to study the properties of string field theory solutions with these mixing properties, one may study a different string field theory containing world-sheet fermions instead of reparametrization ghosts. Namely, string field theory for N = 2 strings [36] shares its internal structure with Berkovits' proposal for a nonpolynomial superstring field theory. Its action and equation of motion include two BRST-like generators G^+ and $\widetilde{G}^+$ (corresponding to the BRST charge Q and the field η_0 from the fermionization of the world-sheet superghosts in the N = 1 case), both mixing world-sheet bosons and fermions (instead of matter and ghost fields for N = 1 strings). The main advantage of this model is its simplicity; no ghosts are needed in addition to the matter fields. The equality of the structure and the simplicity of the field realization of the BRST-like operators turn this theory into a viable candidate for studying the intricacies which general solutions to the equation of motion for nonpolynomial string field theory bring about. Clearly, this equation of motion contains the star product. Whereas the operator formulation [37] and the diagonalization [19, 21, 38] of the bosonic vertex may be transferred literally to N = 2 strings, the fermionic part differs from the N = 1 case since (after twisting, see section 2) the fermions here have conformal weights 0 and 1, respectively. Therefore, we commence the investigation of the operator formulation of the fermionic part of the star product in N = 2 string field theory, which is lacking in the present literature. This also has applications to Berkovits' hybrid formalism [25], where the compactification part of the theory is described by a twisted N = 2 superconformal algebra. Moreover, this $(0, 1)$ first order system is isomorphic to the (ξ, η) ghost system of N = 1 strings. Both, Witten's superstring field theory as well as Berkovits' nonpolynomial string field theory involve this ghost system in a nontrivial manner; the former features insertions of picture changing operators while the latter is formulated in the large Hilbert space. Therefore, a solid understanding of the structure of the star product in this sector

[1] Originally, the discrete version of this map has been proposed in [19] and further developed in [20].

is of interest and can easily be gleaned from the results presented here. Eventually, we point out that this fermionic system is also equivalent to the twisted (b, c) system (cf. end of section 2). In the context of bosonic vacuum string field theory this auxiliary boundary conformal field theory was used to find solutions to the ghost equations of motion from surface state projectors constructed in the twisted (b, c) system [39]. This BCFT is obtained by twisting the energy momentum tensor with the derivative of the ghost number current. The ghost fields of the twisted theory have conformal weights $(1, 0)$ and thus correspond to the fermionic first order system of N = 2 string field theory. However, it is unclear to us whether this equivalence can be traced to any deeper interrelation.

In this paper we construct the vertices needed to formulate N = 2 string field theory in the twisted fermionic sector from scratch using the operator language. In particular we pay attention to the anomaly of the U(1) current J contained in the N = 2 superconformal algebra. Together with the overlap equations for the zero-modes this fixes the choice of vacuum for N-string vertices when one avoids midpoint insertions. For the identity vertex and the reflector the construction is accomplished using δ-function overlap conditions. The reflector is shown to implement BPZ conjugation as a graded antihomomorphism. To obtain the explicit form of the interaction vertex we have to invoke the Neumann function method. Supplemented with the above mentioned conditions on the vacuum the vertex is fixed. The Neumann coefficients are expressed in terms of coefficients of generating functions. We find an intimate relationship between the coefficients for the fermions and those for bosons allowing us to employ known identities from the boson Neumann matrices. Resorting to this relationship we show that the contribution of the $(0, 1)$ system to the reparametrization anomaly cancels those of two real bosons. This is in accordance with their contribution $c = -2$ to the central charge. Finally, we explicitly check that the overlap equations for the interaction vertex are fulfilled.

The paper is organized as follows. In the next section 2 we briefly review the non-polynomial string field theory for N = 2 strings. The embedding of the N = 2 into a small N = 4 superconformal algebra is described and the representation of the generators in terms of N = 2 fields are given. Eventually, this small N = 4 superconformal algebra is twisted, leading to a topological theory [40]. After twisting all fields have integral conformal weights. In section 3 the identity vertex is constructed. As a starting point δ-function overlap conditions for arbitrary N-string vertices are considered. After deducing the form of the identity vertex from the corresponding overlap equations its symmetries are discussed in detail with particular emphasis on the anomaly of the U(1) current. In section 4 the 2-string vertex is considered. Starting from general N-string overlap equations formulated in terms of $\mathbb{Z}_N$-Fourier-transformed fields we discuss constraints on the vacua arising from the zero-mode overlap conditions. Avoiding midpoint insertions these conditions fix the vacuum on which the N-string vertex is built. The reflector is discussed as an application of the tools described in this section. A detailed discussion of BPZ conjugation as implemented by the 2-vertex completes this section. We define BPZ conjugation and its inverse via the the bra-reflector and the ket-reflector, respectively, and their compatibility is shown. The interaction vertex is constructed in section 5. The Neumann coefficients of the 3-string vertex are expressed in terms of generating functions constructed out of the conformal transformations which map unit upper half-disks into the scattering geometry of the vertex. The so obtained Neumann matrices are shown to be closely related to the Neumann matrices for bosons. Therefore, identities for the bosonic Neumann matrices entail corresponding identities for the fermionic ones. In this way, the anomaly of midpoint preserving reparametrizations is shown to cancel the contribution of two real bosons, which is in agreement with conformal field theory arguments. Finally, the overlap conditions for the interaction vertex are checked explicitly. Parts of this calculation are relegated to the appendix where also formulas for the bosonic vertices and Neumann coefficients are collected. The paper is concluded with a short summary and a discussion of possible applications and further developments.

2. Twisting the world-sheet action

Nonpolynomial string field theory for $\mathrm{N}=2$ **strings.** The nonpolynomial string field theory action [36] coincides for $\mathrm{N}=1$ (Neveu-Schwarz) and $\mathrm{N}=2$ strings and contains in its Taylor expansion a kinetic term and a cubic interaction similar to Witten's superstring field theory action. In a notation suitable for both cases, it reads

$$S_{\mathrm{SFT}} = \frac{1}{2g^2} \int \mathrm{tr} \left\{ \left(e^{-\Phi} G^+ e^{\Phi}\right) \left(e^{-\Phi} \widetilde{G}^+ e^{\Phi}\right) - \int_0^1 dt \left(e^{-\widehat{\Phi}} \partial_t e^{\widehat{\Phi}}\right) \left\{ e^{-\widehat{\Phi}} G^+ e^{\widehat{\Phi}}, e^{-\widehat{\Phi}} \widetilde{G}^+ e^{\widehat{\Phi}} \right\} \right\} ; \tag{2.1}$$

here, $e^{\Phi} = \mathcal{I} + \Phi + \frac{1}{2}\Phi \star \Phi + \cdots$ is defined via Witten's midpoint gluing prescription (for convenience, all star products in this action are omitted; $\mathcal{I}$ denotes the identity string field) and Φ is a string field carrying $u(n)$ Chan-Paton labels with an extension $\widehat{\Phi}(t)$ interpolating between $\widehat{\Phi}(t=0)=0$ and $\widehat{\Phi}(t=1)=\Phi$. The BRST-like currents G^+ and $\widetilde{G}^+$ are the two superpartners of the energy-momentum tensor in a twisted small $\mathrm{N}=4$ superconformal algebra with positive U(1)-charge. The action of G^+ and $\widetilde{G}^+$ on any string field is defined in conformal field theory language as taking the contour integral, e.g.

$$\left(G^+ e^{\Phi}\right)(z) = \oint \frac{dw}{2\pi i}\, G^+(w) e^{\Phi}(z)\,, \qquad \left(\widetilde{G}^+ e^{\Phi}\right)(z) = \oint \frac{dw}{2\pi i}\, \widetilde{G}^+(w) e^{\Phi}(z)\,, \tag{2.2}$$

with the integration contour running around z. The corresponding equation of motion reads

$$\widetilde{G}^+ \left(e^{-\Phi} \star G^+ e^{\Phi}\right) = 0\,, \tag{2.3}$$

where contour integrations are implied again. We set out to define unambiguously the star product and the integration symbol in (2.1).

Small $\mathrm{N}=4$ **superconformal algebra.** Just as in bosonic and in superstring theory, the critical dimension for $\mathrm{N}=2$ string theory can be determined from anomaly considerations. It turns out that such a theory has propagating on-shell degrees of freedom only in signature (2,2); thus, the (Kähler) spacetime is naturally parametrized by two complex bosons Z^a,

$$Z^0 := X^1 + \mathrm{i}X^2\,, \qquad Z^1 := X^3 + \mathrm{i}X^4\,. \tag{2.4}$$

JHEP07(2003)061

Their complex conjugates are denoted by $\bar{Z}^{\bar{a}}, \bar{a} \in \{\bar{0}, \bar{1}\}$. The metric on flat $\mathbb{C}^{1,1}$ is taken to be $(\eta_{a\bar{a}})$ with nonvanishing components $\eta_{1\bar{1}} = -\eta_{0\bar{0}} = 1$. To obtain N = 2 world-sheet supersymmetry, the four real bosons have to be supplemented by four Dirac spinors ψ^μ which may be combined into

$$\psi^{+0} := \psi^1 + \mathrm{i}\psi^2 \,, \qquad \psi^{+1} := \psi^3 + \mathrm{i}\psi^4 \,, \qquad \psi^{-\bar{0}} := \psi^1 - \mathrm{i}\psi^2 \,, \qquad \psi^{-\bar{1}} := \psi^3 - \mathrm{i}\psi^4 \,. \tag{2.5}$$

For open strings, we apply the doubling trick throughout this paper so that all fields are (holomorphically) defined on the double cover of the disk, i.e., on the sphere. In superconformal gauge, the world-sheet action for the matter fields now reads (on a euclidean world-sheet Σ with double cover $\tilde{\Sigma}$)

$$S = \frac{1}{4\pi\alpha'} \int_{\tilde{\Sigma}} dz \wedge d\bar{z} \, \left(\partial Z \cdot \bar{\partial} \bar{Z} + \bar{\partial} Z \cdot \partial \bar{Z}\right) + \frac{1}{8\pi} \int_{\tilde{\Sigma}} dz \wedge d\bar{z} \, \left(\psi^+ \cdot \bar{\partial} \psi^- + \psi^- \cdot \bar{\partial} \psi^+\right) . \tag{2.6}$$

The action is normalized in such a way that the operator product expansions are the ones which should be expected from the transition from real to complex coordinates:

$$Z^a(z) \bar{Z}^{\bar{a}}(w) \sim -\alpha' \eta^{a\bar{a}} \ln |z-w|^2 \,, \qquad \psi^{+a}(z) \psi^{-\bar{a}}(w) \sim \frac{2\eta^{a\bar{a}}}{z-w} \,. \tag{2.7}$$

The matter part of the constraint algebra for this theory is an N = 2 superconformal algebra with generators

$$\begin{aligned} T &= -\frac{1}{\alpha'} \partial Z \cdot \partial \bar{Z} - \frac{1}{4} \left(\psi^+ \cdot \partial \psi^- + \psi^- \cdot \partial \psi^+\right) , \\ G^+ &= \frac{\mathrm{i}}{\sqrt{2\alpha'}} \psi^+ \cdot \partial \bar{Z} \,, \qquad G^- = \frac{\mathrm{i}}{\sqrt{2\alpha'}} \psi^- \cdot \partial Z \,, \\ J &= \frac{1}{2} \psi^+ \cdot \psi^- \,. \end{aligned} \tag{2.8}$$

In $D = 4$, the central charge is $c = 6$ (as required from the ghosts, which we will, however, not introduce). Note that the superscripts $\pm$ on each quantity label the charge under the U(1) current J. These currents can, in principle, be defined on general D-dimensional Kähler manifolds for any $D \in 2\mathbb{N}$.

In $D = 4$, we can extend the N = 2 superconformal into a small N = 4 superconformal algebra with additional generators[2] [41]

$$\begin{aligned} J^{++} &= \frac{1}{4} \varepsilon_{ab} \, \psi^{+a} \psi^{+b} \,, \qquad & J^{--} &= \frac{1}{4} \varepsilon_{\bar{a}\bar{b}} \, \psi^{-\bar{a}} \psi^{-\bar{b}} \,, \\ \widetilde{G}^+ &= \frac{\mathrm{i}}{\sqrt{2\alpha'}} \varepsilon_{ab} \, \psi^{+a} \partial Z^b \,, \qquad & \widetilde{G}^- &= \frac{\mathrm{i}}{\sqrt{2\alpha'}} \varepsilon_{\bar{a}\bar{b}} \, \psi^{-\bar{a}} \partial \bar{Z}^{\bar{b}} \,, \end{aligned} \tag{2.9}$$

using the constant antisymmetric tensor ε with $\varepsilon_{01} = \varepsilon_{\bar{0}\bar{1}} = -\varepsilon^{01} = -\varepsilon^{\bar{0}\bar{1}} = 1$. The currents J, J^{++} and J^{--} form an affine $su(1,1)$ Kac-Moody algebra of level 2.

[2] On a D-dimensional manifold, ε_{ab} and $\varepsilon_{\bar{a}\bar{b}}$ have to be replaced by (the components of) nondegenerate $(2,0)$- and $(0,2)$-forms, respectively. (Untwisted) N = 4 supersymmetry requires a hyperkähler spacetime manifold.

In order to obtain an algebra with central charge zero, we can twist the small N $=4$ superconformal algebra by shifting $T \to T' := T + \frac{1}{2}\partial J$, i.e., *reducing* the weight of a field by one half of its charge. After twisting, all fields will have integral weights; in particular, G^+ and $\widetilde{G}^+$ as fields of spin 1 may subsequently serve as BRST-like currents. We will show later that the bosonic and fermionic contributions to the anomalies of all currents in the small N $=4$ superconformal algebra on N-vertices cancel for all even D. Of course, the definition of ε_{ab} requires at least two complex dimensions, i.e., $D \geq 4$.

Twisted action. With respect to the twisted energy-momentum tensor T', ψ^{+a} and $\psi^{-\bar{a}}$ have weights 0 and 1; this suggests that they are no longer complex conjugates in the sense of eq. (2.5). Indeed, they constitute a first order system with euclidean world-sheet action

$$S'_\psi = \frac{1}{4\pi}\int_{\tilde{\Sigma}} dz \wedge d\bar{z}\, \psi^+ \cdot \bar{\partial}\psi^- \tag{2.10}$$

which is real after a Wick back-rotation to Minkowski space for *hermitean* fields $\psi^{\pm}$.[3] That this action for the fermionic part of the twisted theory is indeed the correct one is corroborated by the fact that the full action is invariant under the symmetries generated by all currents in the small twisted N $=4$ superconformal algebra.

As fields of integral weight, both ψ^+ and ψ^- are integer-moded. In particular, the spin 0 field ψ^+ has a zero-mode on the sphere. In analogy to the bc-system there are thus two vacua at the same energy level: the bosonic $\mathrm{SL}(2,\mathbb{R})$-invariant vacuum $|0\rangle =: |{\downarrow}\rangle$ is annihilated by the Virasoro modes $L_{m\geq -1}$ and $\psi^+_{m>0}$, $\psi^-_{m\geq 0}$; its fermionic partner, $|{\uparrow}\rangle := \psi^+_0|{\downarrow}\rangle$, is annihilated by $\psi^+_{m\geq 0}$, $\psi^-_{m>0}$. To get nonvanishing fermionic correlation functions, we need one ψ^+-insertion, i.e., $\langle{\downarrow}|{\downarrow}\rangle = \langle{\uparrow}|{\uparrow}\rangle = 0$, $\langle{\downarrow}|{\uparrow}\rangle = 1$. Taking into account the odd background charge we assign to *dual* vacua (e.g., $|{\uparrow}\rangle$ and $\langle{\downarrow}|$) the same Grassmannality.

Translation to the $\eta\xi$ and twisted bc system. All methods to construct string field theory vertices used in this paper can be formulated in terms of conformal field theory data. In particular, they only depend on the conformal weights and the world-sheet statistics of the fields. The $\psi^-\psi^+$ system as well as the $\eta\xi$ system from the fermionization of the world-sheet superghosts and the twisted bc system of [39] are fermionic first order systems with fields of conformal weights 1 and 0. Therefore, all formulas in this paper remain valid upon the substitutions[4]

$$\begin{array}{ccccc} \psi^+ & \longleftrightarrow & \sqrt{2}\,c' & \longleftrightarrow & \sqrt{2}\,\xi\,, \\ \psi^- & \longleftrightarrow & \sqrt{2}\,b' & \longleftrightarrow & \sqrt{2}\,\eta\,. \end{array} \tag{2.11}$$

The factors of $\sqrt{2}$ take care of the unusual normalization of the $\psi^-\psi^+$ two-point function, which can be read off from eq. (2.7). For example, eq. (2.10) translates via the dictionary given above into the action for the $\eta\xi$ system . Furthermore, states built from $|{\downarrow}\rangle$ are in the small Hilbert space whereas states constructed from $|{\uparrow}\rangle$ are in the large Hilbert space of [42].

[3]Here and in the following, we sometimes omit the spacetime labels on $\psi^{\pm}$ if the statement refers to any of the $\psi^{+a}, \psi^{-\bar{a}}$.

[4]These substitutions can be used to compare part of our results on the 3-vertex with those obtained in [64], a preprint which appeared on the same day.

JHEP07(2003)061

3. Identity vertex

The identity vertex defines the integration in eq. (2.1); it is an element $|\mathcal{I}\rangle$ of the one-string Hilbert space corresponding to the identity string field $\mathcal{I}$. The identity vertex glues the left and right halves of a string together; therefore it can be defined via the corresponding overlap equations.

Overlap equations. In general, the overlap equations for an N-vertex can be determined from conformal field theory arguments [43]: On the world-sheet of the r-th string ($r \in \{1, \ldots, N\}$), a strip, we introduce coordinates $\xi_r = \tau_r + \mathrm{i}\sigma_r$. The strip can be mapped into an upper half-disk with coordinates $z_r = e^{\xi_r}$; the upper half-disks are then glued together in the scattering geometry in a such a way that

$$z_r z_{r-1} = -1 \qquad \text{for } |z_r| = 1\,, \quad \mathrm{Re}(z_r) \geq 0\,, \quad \text{i.e.,} \quad 0 \leq \sigma_r \leq \frac{\pi}{2}\,, \quad \tau_r = 0\,. \tag{3.1}$$

This is achieved by the conformal map

$$f_r(z) = -e^{\mathrm{i}\pi\frac{1-2r}{N}} f(z)\,, \qquad f(z) = \left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right)^{2/N}\,, \tag{3.2}$$

where the phases have been chosen so as to give a symmetric configuration when mapping back to the upper half-plane.

A primary field $\phi^{(r)}$ of conformal weight h in the boundary conformal field theory on the strip is glued according to

$$\begin{aligned}
\phi^{(r)}(\sigma_r, \tau_r = 0) &\equiv \phi^{(r)}(\xi_r) = z_r^h \phi^{(r)}(z_r) \\
&= \left(z_r \frac{\partial z_{r-1}}{\partial z_r}\right)^h \phi^{(r-1)}(z_{r-1}) = \left(\frac{z_r}{z_{r-1}} \frac{\partial z_{r-1}}{\partial z_r}\right)^h \phi^{(r-1)}(\sigma_{r-1}, \tau_{r-1} = 0) \\
&= (-1)^h \phi^{(r-1)}(\pi - \sigma_r, \tau_r = 0) \qquad \text{for } 0 \leq \sigma_r \leq \frac{\pi}{2}\,.
\end{aligned} \tag{3.3}$$

In the last two lines we have used (3.1). This equality is required to hold when applied to the N-string vertex $\langle V_N|$. If we insert the open string mode expansion for $\tau = 0$, $\phi^{(r)}(\sigma) = \phi_0^{(r)} + \sum_n (\phi_n^{(r)} + \phi_{-n}^{(r)}) \cos n\sigma$, we obtain a condition on the modes. For $N \leq 2$, the above condition extends to $0 \leq \sigma \leq \pi$, so that one can take advantage of the orthogonality of the cosine to obtain the diagonal condition $\langle V_N|(\phi_n^{(r)} + \phi_{-n}^{(r)} + (-1)^{n+h}(\phi_{-n}^{(r-1)} + \phi_n^{(r-1)})) = 0$. Instead, we will impose the stricter condition $\langle V_N|(\phi_n^{(r)} + (-1)^{n+h}\phi_{-n}^{(r-1)})) = 0$. For $N > 2$, the overlap equations in general mix all modes.

Construction of the identity vertex. For the $\psi^{\pm}$-system, we demand the stricter conditions

$$\begin{aligned}
\langle \mathcal{I}|\left[\psi_n^+ - (-1)^n \psi_{-n}^+\right] = 0 \quad &\Longrightarrow \quad \langle \mathcal{I}|\psi^+(\sigma) = \langle \mathcal{I}|\psi^+(\pi - \sigma)\,, \\
\langle \mathcal{I}|\left[\psi_n^- + (-1)^n \psi_{-n}^-\right] = 0 \quad &\Longrightarrow \quad \langle \mathcal{I}|\psi^-(\sigma) = -\langle \mathcal{I}|\psi^-(\pi - \sigma)\,,
\end{aligned} \tag{3.4}$$

from which the gluing conditions (3.3) follow. The conditions on $\langle\mathcal{I}|$ are compatible since $\{\psi_n^+ - (-1)^n\psi_{-n}^+, \psi_n^- + (-1)^n\psi_{-n}^-\} = 0$. The obvious solution to eqs. (3.4) reads

$$\begin{aligned}\langle\mathcal{I}| &= \langle\downarrow|\prod_{n=1}^{\infty}\frac{1}{2}\left[\psi_n^+ - (-1)^n\psi_{-n}^+\right]\left[(-1)^n\psi_n^- + \psi_{-n}^-\right]\\ &= \langle\downarrow|\prod_{n=1}^{\infty}\left[1+\frac{1}{2}(-1)^n\psi_n^+\psi_n^-\right] = \langle\downarrow|\exp\left[\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\psi_n^+\psi_n^-\right],\end{aligned} \tag{3.5}$$

where the SL$(2,\mathbb{R})$-invariant vacuum $\langle\downarrow|$ is annihilated by ψ_0^-.

Symmetries of the vertex. Applying the gluing conditions (3.3) to the complex spin 1 fields

$$\partial Z = -\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_k \alpha_k z^{-k-1} \qquad \text{and} \qquad \partial\bar{Z} = -\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_k \bar{\alpha}_k z^{-k-1}, \tag{3.6}$$

we obtain

$$\langle\mathcal{I}|(\alpha_n + (-1)^n\alpha_{-n}) = 0, \qquad \langle\mathcal{I}|(\bar{\alpha}_n + (-1)^n\bar{\alpha}_{-n}) = 0. \tag{3.7}$$

Together with (3.4), this entails that the gluing conditions for the BRST-like spin 1 currents G^+ and $\widetilde{G}^+$,

$$\langle\mathcal{I}|(G_n^+ + (-1)^n G_{-n}^+) = 0, \qquad \langle\mathcal{I}|(\tilde{G}_n^+ + (-1)^n\tilde{G}_{-n}^+) = 0, \tag{3.8}$$

are satisfied. In general, anomalies can only appear if the current contains pairs of conjugate oscillators. Thus, it is clear that the spin 2 currents J^{--}, G^- and $\widetilde{G}^-$ are anomaly-free, just like the spin 0 current J^{++}. More interesting are the (twisted) energy-momentum tensor and the U(1) current J (when treated as primary fields).

The modes of the twisted energy-momentum tensor $T' = -\frac{1}{\alpha'}\partial Z\cdot\partial\bar{Z} - \frac{1}{2}\psi^-\cdot\partial\psi^+$ can be written as

$$L_n = \frac{1}{2}\sum_m \alpha_m\cdot\bar{\alpha}_{n-m} + \frac{1}{2}\sum_m (n-m)\psi_m^-\cdot\psi_{n-m}^+. \tag{3.9}$$

According to (3.3) these modes have to satisfy[5]

$$\langle\mathcal{I}|K_n := \langle\mathcal{I}|(L_n - (-1)^n L_{-n}) = 0 \tag{3.10}$$

for the vertex to be reparametrization invariant. In $D/2$ complex dimensions, the contribution of the bosons to the left hand side of eq. (3.10) can be easily shown to be

$$\langle\mathcal{I}|K_{2n}^{\alpha} = \frac{D}{2}(-1)^n n\,\langle\mathcal{I}|, \tag{3.11}$$

which is canceled by the fermionic contribution

$$\langle\mathcal{I}|K_{2n}^{\psi} = -\frac{D}{2}(-1)^n n\,\langle\mathcal{I}|. \tag{3.12}$$

These contributions arise from terms $\frac{1}{2}\alpha_n\cdot\bar{\alpha}_n$ and $\frac{n}{2}\psi_n^-\cdot\psi_n^+$ in K_{2n}^{α} and K_{2n}^{ψ}, respectively. Due to the absence of such terms, the K_{2n+1} are automatically anomaly-free.

[5]Treating the energy-momentum tensor as a primary field is justified iff the central charge vanishes. In this sense, one can understand eq. (3.10) as a condition on the central charge. Note that the N-string variant $\langle V_N|\sum_{r=1}^{N}(L_n^{(r)} - (-1)^n L_{-n}^{(r)}) = 0$ of eq. (3.10) does not follow from eq. (3.3) for $N > 2$.

Before considering the U(1) current J, let us first recall the discussion in [1] of the U(1)-anomaly of N-vertices: If the current J is bosonized as $J = \partial\varphi$, the action for this boson reads

$$S = -\frac{1}{4\pi}\int dz \wedge d\bar{z}\,(\partial\varphi\bar{\partial}\varphi + QR\varphi)\,. \tag{3.13}$$

The operator product expansion is that of the free action, $\varphi(z)\varphi(w) \sim \ln(z-w)$. The energy-momentum tensor for φ reads $T_\varphi = \frac{1}{2}J^2 - Q\partial J$, where Q is the background charge, i.e., the coefficient of the third order pole in the operator product expansion $T(z)J(w)$. For the $\psi^+\psi^-$-system in $D/2$ complex dimensions, $Q = -D/2$.

In a general gluing geometry the curvature is concentrated in one point, namely the midpoint of the string ($\sigma = \pi/2$). On such surfaces the term linear in φ contributes an anomalous factor of

$$\exp\left(\frac{Q}{2\pi}\varphi\left(\frac{\pi}{2}\right)\int d^2\sigma R\right) \tag{3.14}$$

in the path integral.[6] This integral measures the deficit angle of this surface when circumnavigating the curvature singularity at the string midpoint and contributes $-(N-2)\pi$ for an N-string vertex. Hence, the factor (3.14) produces a U(1)-anomaly of $(N-2)\frac{D}{4}$ in the path integral. Since the U(1)-charge[7] of $\langle\downarrow|$ is $-\frac{D}{4}$, an N-vertex constructed from N $\langle\downarrow|$-vacua requires $(N-2)\frac{D}{4} - N\left(-\frac{D}{4}\right) = \frac{D}{2}(N-1)$ ψ^+-insertions (the exponential factor is neutral).[8] This is consistent with (3.5) for $N=1$.

Therefore, we do not expect the U(1)-current J to be anomaly-free;

$$\langle\mathcal{I}|(J_n + (-1)^n J_{-n}) \neq 0 \tag{3.15}$$

in general. Since its zero-mode measures the fermion number of the vertex, we instead expect $\langle\mathcal{I}|J_0 = \frac{D}{4}\langle\mathcal{I}|$. This relation holds trivially. For $n \neq 0$ in eq. (3.15), one obtains $\langle\mathcal{I}|(J_{2n} + J_{-2n}) = (-1)^n\frac{D}{2}\langle\mathcal{I}|$.

4. Reflector

In this section we construct the 2-string vertex for the fermionic $(1,0)$ system (ψ^-, ψ^+). It is convenient to introduce $\mathbb{Z}_N$-Fourier-transformed fields as a tool to diagonalize general N-string overlap equations. The overlap equations fix the zero-mode part of the 2-vertex up to a sign. A discussion of BPZ conjugation motivates our choice for this sign.

$\mathbb{Z}_N$-transforms. Introducing the combinations

$$\sum_n \psi_n^-\, e^{\pm in\sigma} = \pi_{\psi^+}(\sigma) \pm \mathrm{i}\,\psi^-(\sigma)\,, \qquad \sum_n \psi_n^+\, e^{\pm in\sigma} = \psi^+(\sigma) \pm \mathrm{i}\,\pi_{\psi^-}(\sigma) \tag{4.1}$$

[6] In the integral of the Ricci scalar over this surface we have used $dz \wedge d\bar{z} = 2d\sigma d\tau$.

[7] The U(1) charge of a bra vector is measured by $J_0^\dagger$; we use conventions where $J_0 = \frac{1}{2}\sum_{m=1}^\infty(\psi_{-m}^+\cdot\psi_m^- - \psi_{-m}^-\cdot\psi_m^+) + \frac{1}{2}\psi_0^+\cdot\psi_0^- - \frac{D}{4} = -J_0^\dagger$.

[8] Alternatively, we could use midpoint insertions to adjust the anomaly of the vertex. However, in our cases, they make life unnecessarily complicated. Since the value of the anomaly, $(N-2)\frac{D}{4}$, never exceeds the maximal charge of $\langle\Omega_N|$, i.e. $N\frac{D}{4}$ for $\langle\Omega_N|_{\max} = \langle\uparrow|^{\otimes N}$, we can avoid midpoint insertions.

of left and right movers, the conditions imposed on the fermions following from the δ-function overlap of N strings are

$$\psi^{+(r)}(\sigma) = \begin{cases} \psi^{+(r-1)}(\pi-\sigma), & \sigma \in \left[0\,, \frac{\pi}{2}\right], \\ \psi^{+(r+1)}(\pi-\sigma), & \sigma \in \left[\frac{\pi}{2}, \pi\right], \end{cases}$$

$$\pi^{(r)}_{\psi^+}(\sigma) = \begin{cases} -\pi^{(r-1)}_{\psi^+}(\pi-\sigma), & \sigma \in \left[0\,, \frac{\pi}{2}\right], \\ -\pi^{(r+1)}_{\psi^+}(\pi-\sigma), & \sigma \in \left[\frac{\pi}{2}, \pi\right]. \end{cases} \tag{4.2}$$

For $\psi^-(\sigma)$ and $\pi_{\psi^-}(\sigma)$ similar equations have to be fulfilled. The conditions (4.2) are easily diagonalized if we introduce $\mathbb{Z}_N$-Fourier-transformed fields [37],

$$\Psi^a(\sigma) = \frac{1}{\sqrt{N}} \sum_{r=1}^{N} \psi^{+(r)}(\sigma) e^{\frac{2\pi i r a}{N}}\,, \qquad \Pi^a(\sigma) = \frac{1}{\sqrt{N}} \sum_{r=1}^{N} \pi^{(r)}_{\psi^+}(\sigma) e^{\frac{2\pi i r a}{N}}\,, \tag{4.3}$$

where $a \in \{1, \ldots, N\}$. Note that now (Ψ^a, Π^{N-a}) form canonically conjugate pairs (the upper index is taken modulo N). We choose the following ansatz for the N-vertex in terms of $\mathbb{Z}_N$-transformed oscillators

$$\langle V_N| = \langle \Omega_N| \exp\left(\frac{1}{2} \sum_{a=1}^{N} \sum_{m,n} \Psi^a_m V^a_{mn} \Pi^{N-a}_n\right) \tag{4.4}$$

with Neumann matrices V^a and a vacuum state $\langle \Omega_N|$. The vacuum state will be determined below from the zero-mode overlap conditions; the summation range of m, n should then be adjusted in such a way that only creation operators w.r.t. this vacuum appear in the exponential.

In application to $\langle V_N|$, eqs. (4.2) now read

$$\langle V_N|\Psi^a(\sigma) = \begin{cases} e^{\frac{2\pi i a}{N}} \langle V_N|\Psi^a(\pi-\sigma)\,, \\ e^{\frac{-2\pi i a}{N}} \langle V_N|\Psi^a(\pi-\sigma)\,, \end{cases}$$

$$\langle V_N|\Pi^a(\sigma) = \begin{cases} -e^{\frac{2\pi i a}{N}} \langle V_N|\Pi^a(\pi-\sigma)\,, \\ -e^{\frac{-2\pi i a}{N}} \langle V_N|\Pi^a(\pi-\sigma)\,. \end{cases} \tag{4.5}$$

As already discussed in section 3, the overlap conditions will only contain a sum of two oscillators (rather than infinitely many), if after inserting the mode expansions the cosines can be integrated over $[\,0, \pi]$. This is obviously possible also for $N > 2$ if $\frac{2a}{N} \in \mathbb{N}$. Therefore, (Ψ^N, Π^N) and, if N is even, $(\Psi^{N/2}, \Pi^{N/2})$ appear in the vertex (4.4) with Neumann matrices $V^N = -C$ and $V^{N/2} = C$, respectively. Here, C denotes the twist matrix with components $C_{mn} = (-1)^m \delta_{mn}$.

Before we turn to the 2-string vertex, let us briefly discuss the overlap conditions for the zero-modes of the $\mathbb{Z}_N$-transformed oscillators. It is consistent with (4.5) to demand

$$\langle \Omega_N|\Psi^a_0 = 0 \qquad \text{for } 1 \leq a \leq N-1\,, \tag{4.6a}$$

$$\langle \Omega_N|\Pi^N_0 = 0\,. \tag{4.6b}$$

Note that eqs. (4.6a) entail that no Ψ_0^a (for $a \in \{1,\ldots,N-1\}$) may appear in the exponential of the vertex (4.4). The appearance of Ψ_0^N is forbidden by eq. (4.6b) since $V^N = -C$ is diagonal. In terms of the original one-string oscillators, this means that no ψ_0^+ occurs in the exponential of the vertex.

It is easy to see that the conditions on the vacuum (4.6) are solved by[9]

$$\langle\Omega_N| = \pm\sum_{k=1}^{N} {}_1\langle\uparrow| \otimes \cdots {}_{k-1}\langle\uparrow| \otimes {}_k\langle\downarrow| \otimes {}_{k+1}\langle\uparrow| \otimes \cdots {}_N\langle\uparrow| \,. \tag{4.7}$$

The subscripts indicate in which string Hilbert space the corresponding vacuum state lives. The vacuum (4.7) already features the U(1) charge required by the J-anomaly, namely $(N-2)\frac{D}{4}$. This choice allows us to avoid midpoint insertions.

Overlap equations for the reflector. Expressed in terms of $\mathbb{Z}_2$-transforms, the overlap conditions for the reflector simply become

$$\begin{aligned} \langle V_2|\Psi^1(\sigma) &= -\langle V_2|\Psi^1(\pi-\sigma)\,, \qquad & \langle V_2|\Psi^2(\sigma) &= \langle V_2|\Psi^2(\pi-\sigma)\,,\\ \langle V_2|\Pi^2(\sigma) &= -\langle V_2|\Pi^2(\pi-\sigma)\,, \qquad & \langle V_2|\Pi^1(\sigma) &= \langle V_2|\Pi^1(\pi-\sigma)\,, \end{aligned} \tag{4.8}$$

which can be rewritten in terms of modes acting on $\langle V_2|$ as

$$\begin{aligned} (\Psi_m^1+\Psi_{-m}^1) &= -(-1)^m(\Psi_m^1+\Psi_{-m}^1)\,, \qquad & (\Psi_m^2+\Psi_{-m}^2) &= (-1)^m(\Psi_m^2+\Psi_{-m}^2)\,,\\ (\Pi_m^1+\Pi_{-m}^1) &= (-1)^m(\Pi_m^1+\Pi_{-m}^1)\,, \qquad & (\Pi_m^2+\Pi_{-m}^2) &= -(-1)^m(\Pi_m^2+\Pi_{-m}^2)\,, \end{aligned} \tag{4.9}$$

for the nonzero-modes. The conditions for the zero-modes read

$$\langle V_2|\Psi_0^1 = 0\,, \qquad \langle V_2|\Pi_0^2 = 0\,. \tag{4.10}$$

The zero-modes Ψ_0^2 and Π_0^1 put no restrictions on the vertex. Along the lines of [37], one finds

$$\begin{aligned} \langle V_2| &= \langle\Omega_2|\exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\left[\Psi_m^2(-1)^m\Pi_m^2 - \Psi_m^1(-1)^m\Pi_m^1\right]\right)\\ &= \langle\Omega_2|\exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\left[\psi_m^{+(1)}(-1)^m\psi_m^{-(2)} + \psi_m^{+(2)}(-1)^m\psi_m^{-(1)}\right]\right) \end{aligned} \tag{4.11}$$

as a solution to eqs. (4.9). Since no zero-modes appear in the vertex, the vacuum $\langle\Omega_2|$ has to be annihilated by Ψ_0^1 and Π_0^2 in order to satisfy eq. (4.10). Thus the vacuum is a *symmetric* combination of up- and down-vacua in the two-string Hilbert space,

$$\langle\Omega_2| = \pm({}_1\langle\uparrow| \otimes {}_2\langle\downarrow| + {}_1\langle\downarrow| \otimes {}_2\langle\uparrow|) =: \pm(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)\,. \tag{4.12}$$

This is consistent with eq. (4.7). In the last expression it is understood that the first entry corresponds to string 1, while the second corresponds to string 2. The overall sign is determined by requiring that $\langle V_2|$ implements BPZ conjugation.

[9] Here one has to use the fact that $\langle\uparrow|$ is Grassmann *even* while $\langle\downarrow|$ is Grassmann *odd*, i.e., the bra-vacua have opposite Grassmannality compared to the corresponding ket-vacua. This is a consequence of the odd background charge (cf. the end of section 2).

BPZ conjugation. On a single field $\phi(z)$ BPZ conjugation acts as $I \circ \phi(z)$ with $I(z) = -1/z$; since I inverts the time direction, it is suggestive that on a product of fields, BPZ conjugation should reverse the order of the fields. This statement will be put on a more solid ground below. The action of BPZ on fields induces an action on states: $\mathrm{bpz}(|\phi\rangle)$ defines the out-state $\langle\phi|$ which is created by $\lim_{z\to\infty}\langle 0| I \circ \phi(z)$. In terms of modes this prescription yields

$$\mathrm{bpz}(\phi_n) = (-1)^{n+h}\phi_{-n} \tag{4.13}$$

for a field of conformal weight h. To fix the choice of vacuum in (4.11), recall that $\langle V_2|$ is an element of the tensor product of two dual string Hilbert spaces $\langle V_2| \in \mathcal{H}^* \otimes \mathcal{H}^*$ and thus induces an odd linear map from $\mathcal{H}$ to $\mathcal{H}^*$, which is nothing but BPZ conjugation [44],

$$\langle V_2|\phi\rangle_1 = {}_2\langle \mathrm{bpz}(\phi)| \,. \tag{4.14}$$

In order to be compatible with the usual definitions of BPZ conjugation, we demand in particular that the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum $|{\downarrow}\rangle$ is mapped into $\langle{\downarrow}|$ under BPZ conjugation. Therefore we fix the vacuum $\langle\Omega_2|$ to be

$$\langle\Omega_2| = {}_1\langle{\uparrow}| \otimes {}_2\langle{\downarrow}| + {}_1\langle{\downarrow}| \otimes {}_2\langle{\uparrow}| =: \langle{\uparrow\downarrow}| + \langle{\downarrow\uparrow}| \,. \tag{4.15}$$

Note that with this choice of vacuum and using eq. (4.14) one finds $\mathrm{bpz}(|{\downarrow}\rangle) = \langle{\downarrow}|$ and $\mathrm{bpz}(|{\uparrow}\rangle) = \langle{\uparrow}|$.

Now consider the corresponding ket state $|V_2\rangle$. Observe that the conformal transformation I maps (τ,σ) to $(-\tau, \pi-\sigma)$. Therefore, the overlap equations for $0 \le \sigma \le \pi/2$ for a field ϕ of conformal weight h_ϕ, $\phi^{(r)}(\sigma) = (-1)^{h_\phi}\phi^{(r-1)}(\pi-\sigma)$, transform into $\phi^{(r)}(\sigma) = (-1)^{-h_\phi}\phi^{(r+1)}(\pi-\sigma)$. This implies that the overlap equations for the $N=1$ and $N=2$ vertices are invariant under BPZ conjugation for fields of integral conformal weight. Indeed, this can be verified for (4.9) using (4.13) on the level of modes, and we can immediately write down the solution

$$\begin{aligned}|V_2\rangle &= \exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\left[\Pi^2_{-m}(-1)^m\Psi^2_{-m} - \Pi^1_{-m}(-1)^m\Psi^1_{-m}\right]\right)|\Omega_2\rangle \\ &= \exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\left[\psi^{-(1)}_{-m}(-1)^m\psi^{+(2)}_{-m} + \psi^{-(2)}_{-m}(-1)^m\psi^{+(1)}_{-m}\right]\right)|\Omega_2\rangle \,.\end{aligned} \tag{4.16}$$

It is easy to see that eqs. (4.9), now taken to act on the ket vertex, are fulfilled. Eventually we have to fix our choice of vacuum. In order to fulfill the zero-mode overlap equations (4.10), $|\Omega_2\rangle$ has to be an *antisymmetric* combination of up and down vacua

$$|\Omega_2\rangle = \pm\left(|{\uparrow}\rangle_1 \otimes |{\downarrow}\rangle_2 - |{\downarrow}\rangle_1 \otimes |{\uparrow}\rangle_2\right) = \pm\left(|{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle\right). \tag{4.17}$$

We fix the overall sign of $|\Omega_2\rangle$ to be a plus sign by requiring

$$\mathrm{bpz}^{-1}(\langle\phi|) := {}_2\langle\phi|V_2\rangle_{12}(-1)^{|\phi|+1} = |\mathrm{bpz}^{-1}(\phi)\rangle_1 \,, \tag{4.18}$$

where $|\phi|$ denotes the Grassmannality of the state $|\phi\rangle$. Moreover one finds

$$\begin{aligned}\mathrm{bpz}(\psi^+_{-k}\psi^-_{-l}|\downarrow\rangle) &= {}_{12}\langle V_2|\psi^{+(1)}_{-k}\psi^{-(1)}_{-l}|\downarrow\rangle_1 = {}_{12}\langle V_2|\psi^{+(2)}_{k}(-1)^k\psi^{-(1)}_{-l}|\downarrow\rangle_1 \\ &= {}_{12}\langle V_2|\psi^{-(1)}_{-l}\psi^{+(2)}_{k}(-1)^k(-1)^{|\psi^+||\psi^-|}|\downarrow\rangle_1 \\ &= {}_2\langle\downarrow|\psi^{-(2)}_{l}\psi^{+(2)}_{k}(-1)^{k+l+1}(-1)^{|\psi^+||\psi^-|}\,, \end{aligned} \tag{4.19}$$

which can be checked using (4.11). Eq. (4.19) is the statement that BPZ conjugation acts as a *graded antihomomorphism* on the algebra of modes. To emphasize the gradation we explicitly kept the sign stemming from the anticommutation of the modes. Note that there is no problem in commuting the modes since after acting on the vertex they belong to different Hilbert spaces, so the only effect is an additional sign. Finally, it is straightforward to check that

$${}_{12}\langle V_2|V_2\rangle_{23} = \big(|\uparrow\rangle_3\,{}_1\langle\downarrow| + |\downarrow\rangle_3\,{}_1\langle\uparrow|\big)\exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\left[\psi^{+(3)}_{-m}\psi^{-(1)}_{m} + \psi^{-(3)}_{-m}\psi^{+(1)}_{m}\right]\right) = {}_3\mathbb{1}_1 \tag{4.20}$$

by using standard coherent state techniques (cf. [45]–[47]) and eq. (4.11). One can then check that $\mathrm{bpz}\circ\mathrm{bpz}^{-1} = \mathrm{bpz}^{-1}\circ\mathrm{bpz} = \mathbb{1}$. This completes the construction of the reflector state from the overlap equations.

5. Interaction vertex

In this section, we set up the Neumann function method [48]–[55] for general N-string vertices, since even in terms of the $\mathbb{Z}_N$-Fourier-transforms the overlap equations are not directly soluble for $N \geq 3$. In the case of the 3-string vertex, the Neumann coefficients are computed explicitly in terms of generating functions. The observation that they are intimately related to the well-known bosonic Neumann coefficients helps us to show that the K_n-anomaly of the (bosonic and fermionic) 3-vertex vanishes in any even dimension D. Furthermore, it will be shown that the 3-vertex for the $\psi^+\psi^-$ system satisfies its overlap equations.

Neumann function method. The Neumann function method is based on the fact that the large time transition amplitude is given by the Neumann function of the scattering geometry under consideration. To find the Fock space representation of the interaction vertex one makes an ansatz quadratic in the oscillators,

$$\langle V_N| = \mathcal{N}_N\langle\Omega_N|\exp\left[\frac{1}{4}\sum_{r,s}\sum_{k,l}\psi^{+(r)}_{k}N^{rs}_{kl}\psi^{-(s)}_{l}\right]\,, \tag{5.1}$$

where $\mathcal{N}_N$ is a normalization factor which is determined below.[10] The sum over the string labels r and s runs from 1 to N and the restrictions on the summation range of the oscillator modes has to be determined from the choice of vacuum $\langle\Omega_N|$ (cf. (4.7)) so that

[10]The factor of 1/4 in the exponent will take care of the nonstandard normalization of the correlation function for the complex fermions.

only creation operators appear in the vertex. As derived in section 4, ψ_0^+ does not occur in the exponential. The normalization factor $\mathcal{N}_N$ is determined by taking the matrix element $\langle V_N|\tilde{\Omega}_N\rangle$ where $|\tilde{\Omega}_N\rangle$ is the dual vacuum $\langle \Omega_N|\tilde{\Omega}_N\rangle = 1$. Since this matrix element corresponds to a ψ^+ one-point function and ψ^+ has conformal weight zero this yields $\mathcal{N}_N = \langle V_N|\tilde{\Omega}_N\rangle = \langle \psi^+\rangle = 1$.

To obtain an explicit expression for the coefficients we look at matrix elements of the form

$$G(z,w) = \langle V_N|\psi^{+(s)}(z)\psi^{-(r)}(w)|\tilde{\Omega}_N\rangle \tag{5.2}$$

and reinterpret the result as a correlation function on the disk (or, thanks to PSL$(2,\mathbb{R})$ invariance, equivalently on the upper half-plane). Note that, in this expression, the J-anomaly has to be saturated in each string separately, i.e., in each Hilbert space we need one ψ_0^+ (which can be attributed to either $\langle \Omega_N|$ or $|\tilde{\Omega}_N\rangle$). Inserting the mode expansions for $\psi^+(z)$ and $\psi^-(w)$ into eq. (5.2), one obtains by virtue of eq. (5.1)

$$G(z,w) = \sum_{mn} z^n w^{m-1} N^{rs}_{mn}\,. \tag{5.3}$$

Following [54], we equate this with

$$G(z,w) = \left\langle f_s \circ \psi^+(z) f_r \circ \psi^-(w) \frac{1}{N}\sum_{i=1}^{N} f_i \circ \psi^+(0)\right\rangle, \tag{5.4}$$

where the sum on the right hand side was chosen to distribute the background charge symmetrically among the N strings. In principle, any other choice of $\psi^+(0)$-insertions is admissible as long as the J-anomaly on the scattering geometry is saturated, i.e., we need a total U(1) charge of $+1$ in the correlation function. The f_r map the unit upper half-disk into the corresponding wedge of the scattering geometry, as defined in (3.2). The pole structure of the correlation function (5.4) is easily evaluated; first order poles arise from $\psi^+\psi^-$-contractions, first order zeros from $\psi^+\psi^+$-contractions. Since the conformal weights of ψ^+ and ψ^- are 0 and 1, respectively, we obtain

$$\left\langle f_s \circ \psi^+(z) f_r \circ \psi^-(w) \frac{1}{N}\sum_{i=1}^{N} f_i \circ \psi^+(0)\right\rangle = \frac{2f_r'(w)}{f_s(z)-f_r(w)}\frac{1}{N}\sum_{i=1}^{N}\frac{f_s(z)-f_i(0)}{f_r(w)-f_i(0)}\,. \tag{5.5}$$

Here the unusual factor of 2 appears due to the normalization of the fermionic correlator. From eqs. (5.2) to (5.5) one readily finds the expression for the Neumann coefficients in terms of contour integrals,

$$N^{rs}_{mn} = \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1}w^m} \frac{2f_r'(w)}{f_s(z)-f_r(w)}\frac{1}{N}\sum_{i=1}^{N}\frac{f_s(z)-f_i(0)}{f_r(w)-f_i(0)}\,. \tag{5.6}$$

Neumann coefficients and generating functions. In this paragraph we work out explicitly the integral formula for the Neumann coefficients for the (bra-)interaction vertex and find expressions in terms of the coefficients of generating functions. The vertex will take the form[11]

$$\langle V_3| = \big(\langle\uparrow\uparrow\downarrow| + \langle\uparrow\downarrow\uparrow| + \langle\downarrow\uparrow\uparrow|\big)\exp\left[\frac{1}{4}\sum_{r,s}\sum_{k=1,l=0}^{\infty}\psi_k^{+(r)}N_{kl}^{rs}\psi_l^{-(s)}\right]. \tag{5.7}$$

The maps involved in (5.6) for $N = 3$ can be gleaned from (3.2),

$$f_i(z) = e^{\frac{2\pi\mathrm{i}}{3}(2-i)}\left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right)^{2/3} = \omega^{2-i}f(z) \tag{5.8}$$

with $\omega = e^{\frac{2\pi\mathrm{i}}{3}}$. Using these maps one can rewrite eq. (5.6) as

$$\begin{aligned} N_{mn}^{rs} &= 2\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{1}{z^{n+1}w^m}\frac{f_r'(w)}{f_s(z)-f_r(w)}\frac{f_s(z)f_r(w)^2-1}{f_r(w)^3-1} \\ &= \oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{1}{z^{n+1}w^{m+1}}\frac{2}{3}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\Big[1+U^{rs}(z,w)+U^{sr}(-z,-w)\Big], \end{aligned} \tag{5.9}$$

where

$$U^{rs}(z,w) = \omega^{(s-r)}\frac{w}{z}\left(\frac{1+\mathrm{i}z}{1+\mathrm{i}w}\right)^2 f(w)f(-z)\,. \tag{5.10}$$

Introducing the generating functions

$$G(z) = \frac{f(z)}{(1+\mathrm{i}z)^2} = \sum_{n=0}^{\infty}G_n z^n\,, \tag{5.11a}$$

$$H(z) = (1+\mathrm{i}z)^2 f(-z) = \sum_{n=0}^{\infty}H_n z^n\,, \tag{5.11b}$$

we can write

$$N_{mn}^{rs} = \frac{2}{3}\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\Big[C_{mn}(z,w)+\omega^{(s-r)}U_{mn}(z,w)+\bar{\omega}^{(s-r)}\bar{U}_{mn}(z,w)\Big]\,, \tag{5.12}$$

with

$$\begin{aligned} C_{mn}(z,w) &= \frac{1}{z^{n+1}w^{m+1}}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\,, \\ U_{mn}(z,w) &= \frac{1}{z^{n+2}w^m}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)G(w)H(z)\,, \\ \bar{U}_{mn}(z,w) &= \frac{1}{z^{n+2}w^m}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)G(-w)H(-z)\,. \end{aligned} \tag{5.13}$$

Performing the contour integrals,[12] one finds the Neumann coefficients in terms of the coefficients of the generating functions,

$$N_{mn}^{rs} = \frac{2}{3}\left(C_{mn}+\omega^{(s-r)}U_{mn}+\bar{\omega}^{(s-r)}\bar{U}_{mn}\right)\,, \tag{5.14}$$

[11]Recall that no $\psi_0^{+(r)}$ appears in the exponent as substantiated in section 4.

[12]Note that one can choose the contour always so that only the poles at zero contribute.

JHEP07(2003)061

where

$$
\begin{aligned}
U_{mn} &= \sum_{k=0}^{n} \left[(-1)^{n+1-k} G_{m-n-2+k} - G_{m+n-k}\right] H_k \,, && (5.15)\\
\bar{U}_{mn} &= (-1)^{m+n} \sum_{k=0}^{n} \left[(-1)^{n+1-k} G_{m-n-2+k} - G_{m+n-k}\right] H_k \,. && (5.16)
\end{aligned}
$$

In these formulas, it is implicitly understood that coefficients with negative index are zero, and, as usual, $C_{mn} = (-1)^m \delta_{mn}$ for $m, n > 0$. Since ψ^+ does not appear in the exponential of the vertex, we require $N^{rs}_{0m} = 0$ for $m \geq 0$. Note that the Neumann coefficients are real since the G_n and H_n are real for n even and purely imaginary for n odd. Obviously, $\bar{U}$ is the complex conjugate of U, and $\bar{U} = CUC$. Eq. (5.14) makes the cyclic symmetry of the vertex manifest.

Recursion relations. To find recursion relations for the generating functions (5.11), we observe that $G(z)$ can be expressed in terms of its derivative:

$$G(z) = -\frac{3}{2}\frac{z^2+1}{3z+\mathrm{i}} G'(z)\,. \qquad (5.17)$$

Inserting the mode expansion, one finds

$$G_k = -\oint \frac{dz}{2\pi\mathrm{i}} \frac{1}{z^{k+1}} \frac{3}{2}\frac{z^2+1}{3z+\mathrm{i}} \frac{\partial}{\partial z} G(z)\,. \qquad (5.18)$$

Partially integrating and evaluating the resulting contour integral leads to the following recursion formula for G_n:

$$G_{k+2} = -\frac{2\mathrm{i}}{3(k+2)} G_{k+1} - G_k\,. \qquad (5.19)$$

Note that this complies with the observation that the G_k are alternately real and imaginary. From (5.19) and the initial condition $G_0 = G(0) = 1$ (and $G_{-1} := 0$), the first coefficients are easily computed to be $G_1 = -2\mathrm{i}/3$, $G_2 = -11/9$, and $G_3 = 76\mathrm{i}/81$.

Similarly, we can use

$$H(z) = \left(-\frac{\mathrm{i}}{6} + \frac{z}{2} + \frac{4/3}{3z+\mathrm{i}}\right) H'(z) \qquad (5.20)$$

to find recursion relations for the H_k,

$$(k+2)H_{k+2} = \frac{2\mathrm{i}}{3} H_{k+1} - (k-2)H_k\,, \qquad (5.21)$$

and with the initial condition $H_0 = H(0) = 1$ (and $H_{-1} := 0$), the first coefficients are found to be $H_1 = 2\mathrm{i}/3$, $H_2 = 7/9$ and $H_3 = 32\mathrm{i}/81$. One readily verifies that

$$\sum_{k=0}^{n} G_k H_{n-k} = 0 \qquad \text{for all } n \in \mathbb{N}\,, \qquad (5.22)$$

since $G(z) = 1/H(z)$.

Relation to bosonic coefficients. Exemplarily, the first few Neumann coefficients N^{11}_{mn} can be computed via eqs. (5.14), (5.16) and the recursion relations (5.19) and (5.21):

$$(N^{11})_{mn} = \begin{pmatrix} \frac{10}{27} & 0 & -\frac{64}{729} & 0 & \frac{832}{19683} & \cdots \\ 0 & -\frac{26}{243} & 0 & \frac{1024}{19683} & 0 & \cdots \\ -\frac{64}{243} & 0 & \frac{1786}{19683} & 0 & -\frac{3008}{59049} & \cdots \\ 0 & \frac{2048}{19683} & 0 & -\frac{10250}{177147} & 0 & \cdots \\ \frac{4160}{19683} & 0 & -\frac{15040}{177147} & 0 & \frac{82330}{1594323} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{mn} . \tag{5.23}$$

The above expression holds for $m,n \geq 1$. This suggests that, for these values of m,n, the Neumann coefficients for the $\psi^+\psi^-$-system agree with those for the bosons in the momentum basis (cf. eq. (A.9)) up to some factor; the same can be checked for all other r,s:

$$N^{rs}_{mn} = 2\sqrt{\frac{m}{n}} V^{rs}_{mn} . \tag{5.24}$$

A posteriori, one can easily find a proof for this relation. Comparing with (A.9) and (5.9), we have to show that

$$\begin{aligned} 2\oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1}w^m} \frac{f_r'(w)}{f_s(z)-f_r(w)} \frac{f_s(z)f_r(w)^2-1}{f_r(w)^3-1} &= \\ = -\frac{2}{n}\oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^n w^m} \frac{f_r'(w)f_s'(z)}{(f_s(z)-f_r(w))^2} &. \end{aligned} \tag{5.25}$$

Since the right hand side can be rewritten as

$$\frac{2}{n}\oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^n w^m} \frac{\partial}{\partial z} \frac{f_r'(w)}{(f_s(z)-f_r(w))} = 2\oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1}w^m} \frac{f_r'(w)}{f_s(z)-f_r(w)} , \tag{5.26}$$

the difference of the left hand and the right hand sides of eq. (5.25) is proportional to

$$\oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^{n+1}w^m} \frac{f_r'(w)f_r(w)^2}{f_r(w)^3-1} . \tag{5.27}$$

This expression vanishes for $n > 0$ due to the absence of poles in the z-contour. This establishes the proof of eq. (5.24).

Properties of the Neumann matrices. In view of the close relation between the fermionic and the bosonic Neumann matrices one immediately obtains identities for N^{rs}_{mn}, $m,n \geq 1$ from the bosonic ones. Defining $CN^{r\,r} =: N$, $CN^{r\,r+1} =: N_+$ and $CN^{r\,r-1} =: N_-$, one finds that N, N^+ and N^- mutually commute and

$$\begin{gathered} N + N_+ + N_- = 2 , \qquad N_+ N_- = N(N-2) , \qquad N^2 + N_+^2 + N_-^2 = 4 , \\ N\,N_+ + N_+ N_- + N_- N = 0 , \qquad N_\pm^2 - N_\pm = N\,N_\mp , \\ C\,N = N\,C , \qquad C\,N_+ = N_-\,C . \end{gathered} \tag{5.28}$$

The proof of eq. (5.24) breaks down for $n = 0$. We have $N^{rs}_{00} = 0$; the Neumann coefficients for the case $n = 0$ and $m > 0$ are given by

$$
\begin{aligned}
N^{r\,r}_{m0} &= \begin{cases} -\dfrac{8\mathrm{i}}{9m} G_{m-1} & \text{for } m \text{ even,} \\ 0 & \text{for } m \text{ odd,} \end{cases} \\
N^{r\,r+1}_{m0} &= \begin{cases} \dfrac{4\mathrm{i}}{9m} G_{m-1} & \text{for } m \text{ even,} \\ -\dfrac{4}{3\sqrt{3}m} G_{m-1} & \text{for } m \text{ odd,} \end{cases} \\
N^{r\,r-1}_{m0} &= \begin{cases} \dfrac{4\mathrm{i}}{9m} G_{m-1} & \text{for } m \text{ even,} \\ \dfrac{4}{3\sqrt{3}m} G_{m-1} & \text{for } m \text{ odd.} \end{cases}
\end{aligned} \tag{5.29}
$$

The indices r, s are cyclic. From this it is obvious that

$$
C_{nm} N^{rt}_{m0} = N^{tr}_{n0}\,, \qquad \sum_t \sum_m N^{rt}_{m0} = \sum_t \sum_m N^{tr}_{m0} = 0\,. \tag{5.30}
$$

Exploiting the fact that the generating function $G(z)$ defined in eq. (5.11a) is proportional to the derivative of $((1 - \mathrm{i}z)/(1 + \mathrm{i}z))^{1/3}$, one can show that the coefficients G_k are related to the coefficients a_n (or equivalently A_n) defined in appendix B [37] via

$$
G_{m-1} = \frac{3}{2} m (-\mathrm{i})^{(m-1)} a_m\,. \tag{5.31}
$$

Evaluating the generating function for the coefficients A_{2n} (cf. eq. (B.3)),

$$
\frac{1}{2}\left[\left(\frac{1-\mathrm{i}\,z}{1+\mathrm{i}\,z}\right)^{1/3} + \left(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\right)^{1/3}\right], \tag{5.32}
$$

at $z = 1$, we obtain

$$
\sum_{m=1}^{\infty} N^{11}_{m0} = \frac{4}{3}\sum_{n=1}^{\infty} A_{2n} = \frac{4}{3}\left(\frac{\sqrt{3}}{2} - 1\right). \tag{5.33}
$$

The contour integral around $z = 0$ computes [56]

$$
\sum_{n=0}^{\infty}(A^2_{2n} - A^2_{2n+1}) = \oint \frac{dz}{2\pi\mathrm{i}}\frac{1}{z}\left(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\right)^{1/3}\left(\frac{1+\mathrm{i}\,\frac{1}{z}}{1-\mathrm{i}\,\frac{1}{z}}\right)^{1/3} = \frac{1}{2}\,, \tag{5.34}
$$

which establishes that

$$
\sum_t \sum_{m=1}^{\infty} N^{1t}_{m0} N^{t1}_{m0} = \frac{8}{3}\left(\sum_{n=0}^{\infty}(A^2_{2n} - A^2_{2n+1}) - 1\right) = -\frac{4}{3}\,. \tag{5.35}
$$

Having at hand fermion Neumann coefficients for the nonzero-modes expressed in terms the boson Neumann coefficients puts us in the position to compute the K_n-anomaly of the fermionic 3-vertex in a very simple way. Similarly, the overlap equations can be checked more easily than with the original expression (5.14). This will be done in the next two paragraphs.

Anomaly of the $\psi^{\pm}$-vertex. We will now demonstrate that the contribution of one $\psi^+\psi^-$-pair to the K_n-anomaly of the 3-vertex cancels the contribution of two real (or one complex) bosons. This agrees with the fact that a $(1,0)$-first order system contributes $c=-2$ to the central charge. Thus, in contrast to bosonic and $N=1$ strings, no restriction on the critical dimension follows from the K_n-anomaly.

Namely, let $\sum_{r=1}^{3} K_m^{(r)\psi} = \sum_{r=1}^{3}\left(L_m^{\psi(r)} - (-1)^m L_{-m}^{\psi(r)}\right)$ act on the 3-vertex (5.7). The only contribution to the c-number anomaly comes from the terms in

$$-(-1)^m L_{-m}^{\psi(r)} = -(-1)^m \frac{1}{2}\sum_k (m-k)\psi_{k-m}^{+(r)}\cdot\psi_{-k}^{-(r)} \tag{5.36}$$

containing two creation operators, i.e., from $\frac{1}{2}\sum_{k=0}^{m-1}(m-k)\psi_{k-m}^{+(r)}\cdot\psi_{-k}^{-(r)}$. The action of $\psi_{k-m}^{+(r)}$ on the bra-vertex pulls down a sum over annihilation operators, and from the interchange of $\psi_{-k}^{-(r)}$ with these creation operators we get a c-number term:

$$\begin{aligned}\langle V_3|K_m^{(3)\psi} &= -(-1)^m\frac{1}{2}\sum_{r=1}^{3}\sum_{k=0}^{m-1}(m-k)\langle V_3|\psi_{k-m}^{+(r)}\cdot\psi_{-k}^{-(r)}+\cdots\\ &= -(-1)^m\frac{1}{4}\sum_{r,s=1}^{3}\sum_{k=0}^{m-1}(m-k)\sum_{l=0}^{\infty}\langle V_3|N_{l,m-k}^{sr}\psi_l^{+(s)}\cdot\psi_{-k}^{-(r)}+\cdots\\ &= -(-1)^m\frac{3\,D}{4}\sum_{k=1}^{m-1}(m-k)N_{k,m-k}^{rr}\langle V_3|\,.\end{aligned} \tag{5.37}$$

In the third equality we have used that $N^{11}=N^{22}=N^{33}$ due to cyclicity and that $N_{0,m-k}^{rr}=0$, i.e., the exponential in the vertex contains no ψ_0^+. The dots indicate terms which do not contribute to the c-number anomaly. From (A.12), this equals twice the negative contribution of one real boson; the total anomaly vanishes if we pair each $\psi^{+a}\psi^{-\bar{a}}$-system with a complex boson field $Z^a,\bar{Z}^{\bar{a}}$ in any even dimension.

Overlap conditions. According to the general method outlined in section 4 we introduce $\mathbb{Z}_3$-Fourier-transforms

$$\begin{aligned}\Psi^a &= \frac{1}{\sqrt{3}}\sum_{r=1}^{3}\psi^{+(r)}\omega^{ra}\,,\\ \Pi^a &= \frac{1}{\sqrt{3}}\sum_{r=1}^{3}\psi^{-(r)}\omega^{ra}\,,\end{aligned} \tag{5.38}$$

where $\omega = e^{\frac{2\pi \mathrm{i}}{3}}$ and the index a runs from 1 to 3. This diagonalizes the overlap equations which then read

$$\begin{aligned}
\langle V_3|\Psi^1(\sigma) \;&=\; \begin{cases} \omega\,\langle V_3|\Psi^1(\pi-\sigma)\,, & \sigma\in\left[0\,,\dfrac{\pi}{2}\right], \\ \bar{\omega}\,\langle V_3|\Psi^1(\pi-\sigma)\,, & \sigma\in\left[\dfrac{\pi}{2},\pi\right], \end{cases} \\
\langle V_3|\Psi^2(\sigma) \;&=\; \begin{cases} \bar{\omega}\,\langle V_3|\Psi^2(\pi-\sigma)\,, & \sigma\in\left[0\,,\dfrac{\pi}{2}\right], \\ \omega\,\langle V_3|\Psi^2(\pi-\sigma)\,, & \sigma\in\left[\dfrac{\pi}{2},\pi\right], \end{cases} \\
\langle V_3|\Psi^3(\sigma) \;&=\; \langle V_3|\Psi^3(\pi-\sigma)\,,
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
\langle V_3|\Pi^1(\sigma) \;&=\; \begin{cases} -\omega\,\langle V_3|\Pi^1(\pi-\sigma)\,, & \sigma\in\left[0\,,\dfrac{\pi}{2}\right], \\ -\bar{\omega}\,\langle V_3|\Pi^1(\pi-\sigma)\,, & \sigma\in\left[\dfrac{\pi}{2},\pi\right], \end{cases} \\
\langle V_3|\Pi^2(\sigma) \;&=\; \begin{cases} -\bar{\omega}\,\langle V_3|\Pi^2(\pi-\sigma)\,, & \sigma\in\left[0\,,\dfrac{\pi}{2}\right], \\ -\omega\,\langle V_3|\Pi^2(\pi-\sigma)\,, & \sigma\in\left[\dfrac{\pi}{2},\pi\right], \end{cases} \\
\langle V_3|\Pi^3(\sigma) \;&=\; -\langle V_3|\Pi^3(\pi-\sigma)\,.
\end{aligned} \tag{5.40}$$

These overlap equations can be written in terms of the Fourier modes of the operator [37]

$$\begin{aligned}
Y(\sigma,\sigma') \;&=\; \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\left[\mathrm{i}\,\Theta\left(\frac{\pi}{2}-\sigma\right)-\mathrm{i}\,\Theta\left(\sigma-\frac{\pi}{2}\right)\right]\right)\delta(\sigma+\sigma'-\pi) \\
&=:\,-\frac{1}{2}C(\sigma,\sigma')+\frac{\sqrt{3}}{2}X(\sigma,\sigma')
\end{aligned} \tag{5.41}$$

as

$$\begin{aligned}
\sum_{l=0}^{\infty}\left(\tilde{E}_{kl}+\frac{1}{2}\tilde{C}_{kl}-\frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)\langle V_3|\tilde{\Psi}^1_l \;&=\; 0\,, \\
\sum_{l=0}^{\infty}\left(\tilde{E}_{kl}+\frac{1}{2}\tilde{C}_{kl}+\frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)\langle V_3|\tilde{\Psi}^2_l \;&=\; 0\,, \\
\sum_{l=0}^{\infty}\left(\tilde{E}_{kl}-\frac{1}{2}\tilde{C}_{kl}+\frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)\langle V_3|\tilde{\Pi}^1_l \;&=\; 0\,, \\
\sum_{l=0}^{\infty}\left(\tilde{E}_{kl}-\frac{1}{2}\tilde{C}_{kl}-\frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)\langle V_3|\tilde{\Pi}^2_l \;&=\; 0\,,
\end{aligned} \tag{5.42}$$

where here and in the following the indices $k,l,j\in\mathbb{N}_0$ while $m,n\in\mathbb{N}$. The matrices $\tilde{E}$ and $\tilde{C}$ are given by

$$\tilde{E}_{kl}=2\,\delta_{0k}\delta_{0l}+\delta_{kl}\,,\qquad \tilde{C}_{kl}=(-1)^k\tilde{E}_{kl}\,. \tag{5.43}$$

JHEP07(2003)061

The matrices $\tilde{X}_{kl}$ can be found in appendix C. The redefined oscillators are $\tilde{\Psi}^a_m = \Psi^a_m + \Psi^a_{-m}$ and $\tilde{\Pi}^a_m = \Pi^a_m + \Pi^a_{-m}$ for the nonzero-modes and $\tilde{\Psi}^a_0 = \Psi^a_0$ and $\tilde{\Pi}^a_0 = \Pi^a_0$ for the zero-modes, respectively.

We make the following ansatz for the interaction vertex in terms of the $\mathbb{Z}_3$-transformed oscillators (recall the range of the indices defined above!)

$$\langle V_3| = \langle \Omega_3| \exp\left[\frac{1}{2}\sum_{m,n} \Psi^3_m C_{mn} \Pi^3_n + \sum_{m,k} (\Psi^2_m \tilde{U}_{mk} \Pi^1_k + \Psi^1_m \tilde{\bar{U}}_{mk} \Pi^2_k)\right]. \tag{5.44}$$

Note that the exponential does not contain any Ψ_0 modes. The vacuum $\langle \Omega_3|$ is given by

$$\langle \Omega_3| = \left\langle \Pi^3_0 = 0, \Psi^1_0 = 0, \Psi^2_0 = 0\right|, \tag{5.45}$$

which in terms of one string Hilbert space vacua is expressed as[13]

$$\langle \Omega_3| = {}_1\langle\uparrow| \otimes {}_2\langle\uparrow| \otimes {}_3\langle\downarrow| + {}_1\langle\downarrow| \otimes {}_2\langle\uparrow| \otimes {}_3\langle\uparrow| + {}_1\langle\uparrow| \otimes {}_2\langle\downarrow| \otimes {}_3\langle\uparrow| \,. \tag{5.46}$$

It is straightforward to see that (5.44) satisfies the overlap equations for the Ψ^3_m's and the Π^3_m's. Comparing the $\mathbb{Z}_3$-transformed version of the interaction vertex with eq. (5.7), one can identify

$$\tilde{U}_{ml} = U_{ml}\,, \qquad \tilde{\bar{U}}_{ml} = \bar{U}_{ml}. \tag{5.47}$$

Inserting the ansatz (5.44) in (5.42), one obtains the overlap equations for the oscillators Π^1, Π^2 and Ψ^1, Ψ^2 in matrix form

$$\sum_{l=0}^{\infty}\left(\tilde{E}_{kl} - \frac{1}{2}\tilde{C}_{kl} + \frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)(\delta_{lj} - U_{lj}) = 0\,, \tag{5.48a}$$

$$\sum_{l=0}^{\infty}\left(\tilde{E}_{kl} - \frac{1}{2}\tilde{C}_{kl} - \frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)(\delta_{lj} - \bar{U}_{lj}) = 0\,, \tag{5.48b}$$

$$\sum_{l=0}^{\infty}\left(\tilde{E}_{kl} + \frac{1}{2}\tilde{C}_{kl} - \frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)(\delta_{lm} + \bar{U}^T_{lm}) = 0\,, \tag{5.48c}$$

$$\sum_{l=0}^{\infty}\left(\tilde{E}_{kl} + \frac{1}{2}\tilde{C}_{kl} + \frac{\sqrt{3}}{2}\tilde{X}_{kl}\right)(\delta_{lm} + U^T_{lm}) = 0\,. \tag{5.48d}$$

Let us now exemplify that these overlap conditions are indeed fulfilled by the matrices given in eq. (5.16). In particular we consider the parts of the overlap equations involving zero-modes.

Consider the $k = 0$ overlap equation for Π^1_0, which is the zero-zero component of eq. (5.48a):

$$1 - \frac{\sqrt{3}}{2}\sum_{m=1}^{\infty} \tilde{X}_{0m} U_{m0} \stackrel{!}{=} 0\,. \tag{5.49}$$

[13]The overlap equations fix the vacuum up to an overall sign factor.

Inserting the U_{m0} component

$$U_{m0} = -(-\mathrm{i})^m \, a_m \tag{5.50}$$

into (5.49) allows us to use known summation formulas for the coefficients [37, 57] to obtain

$$\sum_{m=1}^{\infty} \tilde{X}_{0m} U_{m0} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{a_{2k+1}}{2k+1} = \frac{2}{\sqrt{3}} \tag{5.51}$$

proving eq. (5.49). Consider now the overlap equations for $k \neq 0$. Setting $k = 2l$ for k even and $k = 2l+1$ for k odd yields

$$\begin{aligned} -\frac{1}{2} U_{2l,0} - \frac{\sqrt{3}}{2} \sum_{m=1}^{\infty} \tilde{X}_{2l,m} U_{m0} &\stackrel{!}{=} 0\,, \\ -\frac{3}{2} U_{2l+1,0} + \frac{\sqrt{3}}{2} \tilde{X}_{2l+1,0} - \frac{\sqrt{3}}{2} \sum_{m=1}^{\infty} \tilde{X}_{2l+1,m} U_{m0} &\stackrel{!}{=} 0\,. \end{aligned} \tag{5.52}$$

The first of these equations is proven by

$$\frac{\sqrt{3}}{2} \sum_{m=1}^{\infty} \tilde{X}_{2l,m} U_{m0} = \frac{\sqrt{3}}{\pi} \sum_{k=0}^{\infty} (-1)^l \left(\frac{a_{2k+1}}{2k+1+2l} + \frac{a_{2k+1}}{2k+1-2l} \right) = \frac{1}{2} (-1)^l a_{2l}\,. \tag{5.53}$$

The second equation in (5.52) is fulfilled due to

$$\begin{aligned} \frac{\sqrt{3}}{2} \sum_{m=1}^{\infty} \tilde{X}_{2l+1,m} U_{m0} &= -\frac{\sqrt{3}\mathrm{i}}{\pi} \sum_{k=0}^{\infty} (-1)^l \left(\frac{a_{2k}}{2k+2l+1} - \frac{a_{2k}}{2k-2l-1} \right) + \frac{\sqrt{3}\mathrm{i}}{\pi} (-1)^l \frac{2\,a_0}{2k+1} \\ &= -\frac{3\mathrm{i}}{2} (-1)^l a_{2l+1} + \frac{\sqrt{3}\mathrm{i}}{\pi} (-1)^l \frac{2\,a_0}{2k+1}\,. \end{aligned} \tag{5.54}$$

More involved overlap conditions can be proven using techniques developed in [37, 57]. We postpone their discussion to appendix C.

6. Conclusions

In this paper we explicitly constructed the string field theory vertices for a fermionic first order system $\psi^{\pm}$ with conformal weights $(1, 0)$ in the operator formulation. The technical ingredients needed to construct general N-string vertices are presented in detail. The identity vertex, the reflector and the interaction vertex are discussed with emphasis on their charge under the anomalous U(1) current J and their zero-mode dependence. The identity vertex and the reflector are derived from the corresponding δ-function overlap conditions. The reflector is shown to implement BPZ conjugation as a graded antihomomorphism, and some consistency conditions on the gluing of the reflector are checked. The construction of the interaction vertex is achieved by invoking the Neumann function method. The coefficients of the Neumann matrices are given in terms of coefficients of generating functions and recursion relations for these coefficients are derived. The Neumann coefficients for the $\psi^{\pm}$ system are neatly expressed in terms of those for the bosons. This allows us to infer

JHEP07(2003)061

identities for the fermion Neumann matrices directly from those for the bosons. Moreover, the c-number anomaly of midpoint preserving reparametrizations for a $\psi^{\pm}$ pair is straightforwardly shown to cancel the contribution of two real bosons. This agrees with the fact that a $(0,1)$ first order system contributes $c = -2$ to the central charge. Eventually, it is shown that the overlap equations following from the δ-function overlap conditions are satisfied by the Neumann matrices.

Clearly, the work presented is meant to be a starting point for further investigations. Diagonalizing the vertex is a straightforward task and is attacked in a forthcoming paper [58]. This should pave the way for studying solutions to string field theory in several contexts. Firstly, one might examine how the solution generating techniques derived from integrability [59] and proposed in [41, 60, 61] perform in the more controlled setting of N = 2 SFT. This can be expected to give valuable information about how solutions to string field theory can be constructed dropping the factorization assumption of vacuum string field theory. As a direct application to N = 1 superstring field theory it appears to be worthwhile to investigate the dependence of solutions on the $\eta\xi$-system more closely. This fermionic $(0,1)$ first order system emerges in the bosonization of the superconformal ghosts. The related picture changing operation is a delicate subject in string (field) theory and deserves to be examined with minuteness. Finally, we want to point out the similarity of the fermionic first order system considered in this paper and twisted bc-system. This auxiliary boundary conformal field theory is used in vacuum string field theory to construct solutions to the ghost part of the equation of motion. The solutions to these equations become projectors in the *twisted* theory. It is tempting to speculate that this similarity can be traced to a deeper interrelation.

Acknowledgments

We would like to thank O. Lechtenfeld, A.D. Popov, M. Ihl and M. Wolf for useful discussions. This work was done within the framework of the DFG priority program "String Theory" (SPP 1096).

A. Review of bosonic results

In this section we briefly review the results on bosonic vertices in string field theory which will be useful in section 5. The operator formulation of Witten's open string field theory has been developed in [37, 62, 63] for the bosonic string and in [43] for the NSR superstring.

Bosonic vertices. Using the open string mode expansion and corresponding creation and annihilation operators[14]

$$X(\sigma) = x_0 + \sqrt{2}\sum_{m=1}^{\infty} x_n \cos n\sigma \quad \text{with} \quad x_n = \mathrm{i}\sqrt{\frac{\alpha'}{n}}\left(a_n - a_n^{\dagger}\right), \quad x_0 = \mathrm{i}\sqrt{\frac{\alpha'}{2}}\left(a_0 - a_0^{\dagger}\right), \tag{A.1}$$

[14]Strictly speaking the formulas given in the following are valid for one single boson but straightforwardly generalizable to any number of spacetime directions by introducing the corresponding Lorentz indices.

one obtains the identity vertex[15]

$$X(\sigma) = X(\pi - \sigma) \quad \Rightarrow \quad \langle V_1| = \langle \mathcal{I}| = \langle 0| \exp\left[-\frac{1}{2}\sum_{k,l\geq 0}^{\infty} a_k C_{kl}\, a_k\right], \tag{A.2}$$

where $C_{kl} = (-1)^k \delta_{kl}$. The bra-vacuum used in this expression is the oscillator vacuum which is annihilated by all $a_k^\dagger$. The 2-string overlap conditions

$$X^{(1)}(\sigma) = X^{(2)}(\pi - \sigma) \quad \Rightarrow \quad \langle V_2|\, x_n^{(1)} = (-1)^n \langle V_2|\, x_n^{(2)}, \tag{A.3}$$

are solved in terms of the same matrix C by

$$\langle V_2| = \langle 0| \exp\left[-\frac{1}{2}\sum_{k,l\geq 0}^{\infty} a_k^{(1)} C_{kl}\, a_l^{(2)}\right]. \tag{A.4}$$

The construction of the interaction vertex $\langle V_3|$ is more involved, and the overlap equations are not solved directly. They are most conveniently formulated in terms of $\mathbb{Z}_3$-transformed string oscillators $A_k^{(a)}$ (cf. section 4, not to be confused with the coefficients A_k defined in (B.3)). By using

$$\langle V_3| = \langle 0| \exp\left[-\sum_{k,l\geq 0}^{\infty}\left(\frac{1}{2}A_k^{(3)} C_{kl}\, A_l^{(3)} + A_k^{(1)} U_{kl}\, A_l^{(2)}\right)\right] \tag{A.5}$$

as an ansatz for the vertex, one can write the overlap equations in matrix form as

$$(\mathbb{1} - Y)E(\mathbb{1} + U) = 0\,, \qquad (\mathbb{1} + Y)E^{-1}(\mathbb{1} - U) = 0\,. \tag{A.6}$$

Here the matrix E has components $E_{mn} = \delta_{m,0}\delta_{n,0} + \sqrt{\frac{2}{n}}\delta_{mn}$, and the matrix Y is given by the Fourier components of the operator (5.41). Rewritten in terms of the original one-string oscillators the vertex takes the form[16]

$$\langle V_3| = \langle 0| \exp\left[-\frac{1}{2}\sum_{r,s}\sum_{m,n\geq 0}^{\infty} a_n^{(r)} V'^{rs}_{nm}\, a_m^{(s)}\right] \tag{A.7}$$

with the matrices $(r = 1, 2, 3)$

$$\begin{aligned}
V'^{rr}_{nm} &= \frac{1}{3}(C + U + \bar{U})\,, \\
V'^{r\,r+1}_{nm} &= \frac{1}{3}\left(C - \frac{1}{2}(U + \bar{U}) + \frac{\sqrt{3}}{2}(U - \bar{U})\right), \\
V'^{r\,r-1}_{nm} &= \frac{1}{3}\left(C - \frac{1}{2}(U + \bar{U}) - \frac{\sqrt{3}}{2}(U - \bar{U})\right).
\end{aligned} \tag{A.8}$$

[15]Originally, in [37] the corresponding ket-vector has been constructed. For coherence with the rest of the presentation we review the bosonic results in terms of bra-vectors.

[16]We stick to the notation of [10] and denote the matrices in the oscillator basis with a prime while those in momentum basis are unprimed.

JHEP07(2003)061

After transformation to momentum basis, these matrices can be identified with the Neumann coefficients. The explicit coefficients are listed in appendix B. Following the conformal field theory approach of [54], one can express the Neumann coefficients via contour integrals

$$V_{mn}^{rs} = -\frac{1}{\sqrt{mn}} \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{1}{z^n\, w^m} \frac{f_r'(w) f_s'(z)}{(f_s(z) - f_r(w))^2}\,, \tag{A.9}$$

where the f_i's map the scattering geometry of the interaction vertex to the disk (cf. section 3).

Reparametrization anomaly. Reparametrizations generated by $K_n = L_n - (-1)^n L_{-n}$ leave the midpoint invariant and are classical symmetries of string field theory. However, for nonvanishing central charge one finds an anomaly in these reparametrizations. In the operator formulation this anomaly arises from operator orderings when two creation operators act on the vertex, i.e., from terms like

$$-(-1)^m \frac{1}{2} \sum_{k=1}^{m-1} \sqrt{k(m-k)} a_k^\dagger \cdot a_{m-k}^\dagger \tag{A.10}$$

contained in K_m. In [62] it was shown that a single boson contributes (see also [14])

$$\langle V_3| \left(K_{2n}^{(1)} + K_{2n}^{(2)} + K_{2n}^{(3)} \right) = -\frac{5}{18} n(-1)^n \langle V_3|\,. \tag{A.11}$$

However, due to a nontrivial relation between the Neumann coefficients for the bosons and the fermions in the twisted theory we can show that in the full theory this anomaly is canceled in any (even) dimension. To this end let us derive the contribution of D bosons to the anomaly in terms of the Neumann coefficients. The application of $\sum_{t=1}^{3} K_m^{(t)}$ to $\langle V_3|$ yields

$$\begin{aligned}
\sum_{t=1}^{3} \langle V_3| K_m^{(t)} &= -(-1)^m \frac{1}{2} \sum_{t=1}^{3} \sum_{k=1}^{m-1} \sqrt{k(m-k)} \langle V_3| a_k^{\dagger(t)} \cdot a_{m-k}^{\dagger(t)} + \cdots \\
&= (-1)^m \frac{1}{4} \sum_{t=1}^{3} \sum_{k=1}^{m-1} \sqrt{k(m-k)} \langle V_3| a_n^{(r)} (V_{nk}^{rt} + V_{kn}^{tr}) \cdot a_{m-k}^{\dagger(t)} + \cdots \\
&= (-1)^m \frac{D}{4} \sum_{t=1}^{3} \sum_{k=1}^{m-1} \sqrt{k(m-k)} \langle V_3| (V_{m-k,k}^{tt} + V_{k,m-k}^{tt}) \\
&= (-1)^m \frac{3\,D}{2} \sum_{k=1}^{m-1} \sqrt{k(m-k)}\, V_{k,m-k}^{tt} \langle V_3|\,.
\end{aligned} \tag{A.12}$$

B. Bosonic Neumann coefficients

In this appendix we give a list of the boson Neumann coefficients. These are results of [37] but presented in the notation of [10]. The interaction vertex in momentum basis is given by

$$\langle V_3| = \int d^D p^{(1)} d^D p^{(2)} d^D p^{(3)} \delta^D (p^{(1)} + p^{(2)} + p^{(3)})_{123} \langle p, 0| \exp\left[-V \right], \tag{B.1}$$

where

$$V=\frac{1}{2}\sum_{r,s}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(r)\mu}V_{mn}^{rs}a_n^{(s)\nu}+\sqrt{\alpha'}\sum_{r,s}\sum_{n\geq 1}\eta_{\mu\nu}p^{(r)\mu}V_{0n}^{rs}a_n^{(s)\nu}+\frac{\alpha'}{2}\sum_r\eta_{\mu\nu}p^{(r)\mu}V_{00}^{rr}p^{(r)\nu}\,. \tag{B.2}$$

The Neumann matrices are expressed in terms of coefficients A_n and B_n which are defined as

$$\left(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\right)^{1/3}=\sum_{n\text{ even}}A_n\,z^n+\mathrm{i}\sum_{n\text{ odd}}A_n\,z^n\,,\quad \left(\frac{1+\mathrm{i}\,z}{1-\mathrm{i}\,z}\right)^{2/3}=\sum_{n\text{ even}}B_n\,z^n+\mathrm{i}\sum_{n\text{ odd}}B_n\,z^n\,. \tag{B.3}$$

The Neumann coefficients read

$$\begin{aligned}
V_{mn}^{r\,r} &= -\sqrt{mn}\,\frac{(-1)^n+(-1)^m}{6}\left(\frac{A_m\,B_n+A_n\,B_m}{m+n}+\frac{A_m\,B_n-A_n\,B_m}{m-n}\right),\\
&\quad m\neq n,\ m,n\neq 0\,,\\
V_{mn}^{r\,r+1} &= \sqrt{mn}\,\frac{(-1)^n+(-1)^m}{12}\left(\frac{A_m\,B_n+A_n\,B_m}{m+n}+\frac{A_m\,B_n-A_n\,B_m}{m-n}\right)-\\
&\quad -\sqrt{mn}\,\sqrt{3}\,\frac{1-(-1)^{n+m}}{12}\left(\frac{A_m\,B_n-A_n\,B_m}{m+n}+\frac{A_m\,B_n+A_n\,B_m}{m-n}\right),\\
&\quad m\neq n,\ m,n\neq 0\,,\\
V_{mn}^{r\,r-1} &= \sqrt{mn}\,\frac{(-1)^n+(-1)^m}{12}\left(\frac{A_m\,B_n-A_n\,B_m}{m-n}+\frac{A_m\,B_n+A_n\,B_m}{m+n}\right)+\\
&\quad +\sqrt{mn}\,\sqrt{3}\,\frac{1-(-1)^{n+m}}{12}\left(\frac{A_m\,B_n+A_n\,B_m}{m-n}+\frac{A_m\,B_n-A_n\,B_m}{m+n}\right),\\
&\quad m\neq n,\ m,n\neq 0\,.
\end{aligned} \tag{B.4}$$

The coefficients on the diagonal are given by

$$\begin{aligned}
V_{nn}^{r\,r} &= -\frac{1}{3}\left[2\sum_{k=0}^{n}(-1)^{n-k}A_k^2-(-1)^n-A_n^2\right],\qquad n\neq 0\,,\\
V_{nn}^{r\,r+1} &= V_{nn}^{r,r-1}=\frac{1}{2}\left[(-1)^n-V_{nn}^{rr}\right],\qquad n\neq 0\,,\\
V_{00}^{r\,r} &= \ln\left(\frac{27}{16}\right).
\end{aligned} \tag{B.5}$$

The coefficients with one index zero are obtained as limits of the coefficients V_{mn} above

$$\begin{aligned}
V_{0n}^{r\,r} &= -\sqrt{\frac{2}{n}}\,\frac{1+(-1)^n}{3}A_n\,,\\
V_{0n}^{r\,r+1} &= -\sqrt{\frac{2}{n}}\left[-\frac{1+(-1)^n}{6}A_n-\sqrt{3}\frac{1-(-1)^n}{6}A_n\right],\\
V_{0n}^{r\,r-1} &= -\sqrt{\frac{2}{n}}\left[-\frac{1+(-1)^n}{6}A_n+\sqrt{3}\frac{1-(-1)^n}{6}A_n\right].
\end{aligned} \tag{B.6}$$

The value of the coefficients for different conventions for α' are easily obtained absorbing the explicit α' into these coefficients.

C. More overlap equations

In this appendix we continue the discussion of the overlap equations started at the end of section 5. We adopt the convention about the index range chosen there so that the indices k, l and j start from zero $k, l, j = 0, 1, \ldots$, while $m, n = 1, 2, \ldots$. The matrices $\tilde{X}_{kl}$ are

$$\tilde{X}_{0m} = -\tilde{X}_{m0} = \frac{2\,\mathrm{i}}{\pi\, m}(-1)^{\frac{m-1}{2}}\left[1-(-1)^m\right], \tag{C.1}$$

$$\tilde{X}_{nm} = \frac{\mathrm{i}}{\pi}(-1)^{\frac{n-m-1}{2}}\left[1-(-1)^{n+m}\right]\left[\frac{1}{n+m}+\frac{(-1)^m}{n-m}\right]. \tag{C.2}$$

Note that compared to the matrices defined in [37] we have $\tilde{X}_{nm} = X^{\mathrm{GJ}}_{nm}$ but for the parts containing a zero index $-\sqrt{2}\,\tilde{X}_{0m} = X^{\mathrm{GJ}}_{0m}$! Using the relation to the bosonic coefficients given in eq. (5.24) and the definition of $N^{r,r+1}_{mn}$ in (5.14) one finds for $m \neq n$

$$\begin{aligned} N^{r\,r}_{mn} &= \frac{2}{3}(U_{mn}+\bar{U}_{mn}) = 2\sqrt{\frac{m}{n}}V^{r\,r}_{mn}\,, \\ N^{r\,r+1}_{mn} &= \frac{2}{3}(\omega U_{mn}+\bar{\omega}\bar{U}_{mn}) = -\frac{1}{3}(U_{mn}+\bar{U}_{mn})+\frac{\mathrm{i}}{\sqrt{3}}(U_{mn}-\bar{U}_{mn}) = 2\sqrt{\frac{m}{n}}V^{r\,r+1}_{mn}\,, \end{aligned} \tag{C.3}$$

and hence

$$\begin{aligned} U_{mn} &= -\frac{m}{4}\left[(-1)^n+(-1)^m\right]\left[\frac{A_m\,B_n+A_n\,B_m}{m+n}+\frac{A_m\,B_n-A_n\,B_m}{m-n}\right]+ \\ &\quad+\frac{\mathrm{i}\,m}{4}\left[1-(-1)^{m+n}\right]\left[\frac{A_m\,B_n-A_n\,B_m}{m+n}+\frac{A_m\,B_n+A_n\,B_m}{m-n}\right], \quad m\neq n\,, \end{aligned} \tag{C.4}$$

from which it is once more apparent that $CUC = \bar{U}$. This prepares the stage to scrutinize the overlap equations for Π^1_m following from eq. (5.48a). Taking $k = 0$ and $j = 2l$ in eq. (5.48a), one finds

$$\sum_{m=1}^{\infty}\tilde{X}_{0m}U_{m,2l} \stackrel{!}{=} 0\,. \tag{C.5}$$

Inserting eqs. (C.1) and (C.4) yields

$$\sum_{m=1}^{\infty}\tilde{X}_{0m}U_{m,2l} = -\frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\frac{A_{2k+1}\,B_{2l}-A_{2l}\,B_{2k+1}}{2k+1+2l}+\frac{A_{2k+1}\,B_{2l}+A_{2l}\,B_{2k+1}}{2k+1-2l}\right]. \tag{C.6}$$

Using the relation between the coefficients A_n and a_n,

$$A_{2k} = (-1)^k a_{2k}\,, \qquad A_{2k+1} = (-1)^k a_{2k+1}\,, \tag{C.7}$$

and the summation formulas for the coefficients a_n derived in [37]

$$O_k^a = \sum_{l=0}^{\infty} \frac{a_{2l+1}}{(2l+1)+k} = \frac{\pi\, a_k}{\sqrt{3}}\,, \quad O^a_{-n} = \sum_{l=0}^{\infty} \frac{a_{2l+1}}{(2l+1)-n} = -\frac{1}{2}\frac{\pi\, a_n}{\sqrt{3}}\,, \quad \text{for } k,n \text{ even,} \tag{C.8a}$$

$$E_k^a = \sum_{l=0}^{\infty} \frac{a_{2l}}{(2l)+k} = \frac{\pi\, a_k}{\sqrt{3}}\,, \qquad E^a_{-n} = \sum_{l=0}^{\infty} \frac{a_{2l+1}}{(2l+1)-n} = -\frac{1}{2}\frac{\pi\, a_n}{\sqrt{3}}\,, \quad \text{for } k,n \text{ odd,} \tag{C.8b}$$

$$O_k^b = \sum_{l=0}^{\infty} \frac{b_{2l+1}}{(2l+1)+k} = \frac{\pi\, a_k}{\sqrt{3}}\,, \quad O^b_{-n} = \sum_{l=0}^{\infty} \frac{b_{2l+1}}{(2l+1)-n} = \frac{1}{2}\frac{\pi\, b_n}{\sqrt{3}}\,, \quad \text{for } k,n \text{ even,} \tag{C.8c}$$

$$E_k^b = \sum_{l=0}^{\infty} \frac{b_{2l}}{(2l)+k} = \frac{\pi\, b_k}{\sqrt{3}}\,, \qquad E^b_{-n} = \sum_{l=0}^{\infty} \frac{b_{2l+1}}{(2l+1)-n} = \frac{1}{2}\frac{\pi\, b_n}{\sqrt{3}}\,, \quad \text{for } k,n \text{ odd,} \tag{C.8d}$$

one finds

$$\sum_{m=1}^{\infty} \tilde{X}_{0m} U_{m,2l} = -\frac{2}{\pi}\left[O^a_{2l}B_{2l} - O^b_{2l}A_{2l} + O^a_{-2l}B_{2l} + O^b_{-2l}A_{2l}\right] = 0 \tag{C.9}$$

proving (C.5). Now let us look at $k = 2l+1$ and $j = 2n+1$ in eq. (5.48a):

$$\frac{3}{2}U_{2l+1,2n+1} + \frac{\sqrt{3}}{2}\sum_{m=0}^{\infty} \tilde{X}_{2l+1,m}U_{m,2n+1} \stackrel{!}{=} 0\,. \tag{C.10}$$

By inserting eqs. (C.2) and (C.4), the sum can be written in terms of eqs. (C.8) as

$$\begin{aligned}\sum_{m=0}^{\infty} \tilde{X}_{2l+1,m}U_{m,2n+1} &= (-1)^l\,\frac{2l+1}{\pi} \times \\ &\times \left[\frac{E^a_{2l+1}B_{2n+1} - E^b_{2l+1}A_{2n+1}}{(2n+1)-(2l+1)} + \frac{E^a_{-2l-1}B_{2n+1} - E^b_{-2l-1}A_{2n+1}}{(2n+1)+(2l+1)} - \right. \\ &\quad \left. - \frac{E^b_{-2l-1}A_{2n+1} + E^a_{-2l-1}B_{2l+1}}{(2n+1)-(2l+1)} - \frac{E^a_{2l+1}B_{2n+1} + E^b_{2l+1}A_{2n+1}}{(2n+1)+(2l+1)}\right] \\ &= -\sqrt{3}\,U_{2l+1,2n+1}\,,\end{aligned} \tag{C.11}$$

proving eq. (C.10). The case $k = 2l$ and $j = 2n$ can be easily treated along the same lines.

References

[1] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[2] A. Sen, *Descent relations among bosonic D-branes*, *Int. J. Mod. Phys.* **A 14** (1999) 4061 [hep-th/9902105].

[3] A. Sen, *Non-BPS states and branes in string theory*, hep-th/9904207.

[4] A. Sen, *Universality of the tachyon potential*, *J. High Energy Phys.* **12** (1999) 027 [hep-th/9911116].

[5] K. Ohmori, *A review on tachyon condensation in open string field theories*, hep-th/0102085.

[6] P.-J. De Smet, *Tachyon condensation: calculations in string field theory*, hep-th/0109182.

[7] I.Ya. Aref'eva, D.M. Belov, A.A. Giryavets, A.S. Koshelev and P.B. Medvedev, *Noncommutative field theories and (super)string field theories*, hep-th/0111208.

[8] W. Taylor, *Lectures on D-branes, tachyon condensation and string field theory*, hep-th/0301094.

[9] L. Rastelli, A. Sen and B. Zwiebach, *String field theory around the tachyon vacuum*, *Adv. Theor. Math. Phys.* **5** (2002) 353 [hep-th/0012251].

[10] L. Rastelli, A. Sen and B. Zwiebach, *Classical solutions in string field theory around the tachyon vacuum*, *Adv. Theor. Math. Phys.* **5** (2002) 393 [hep-th/0102112].

[11] L. Rastelli, A. Sen and B. Zwiebach, *Half strings, projectors and multiple D-branes in vacuum string field theory*, *J. High Energy Phys.* **11** (2001) 035 [hep-th/0105058].

[12] D.J. Gross and W. Taylor, *Split string field theory, 1*, *J. High Energy Phys.* **08** (2001) 009 [hep-th/0105059].

[13] D.J. Gross and W. Taylor, *Split string field theory, 2*, *J. High Energy Phys.* **08** (2001) 010 [hep-th/0106036].

[14] M. Schnabl, *Anomalous reparametrizations and butterfly states in string field theory*, *Nucl. Phys.* **B 649** (2003) 101 [hep-th/0202139].

[15] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Star algebra projectors*, *J. High Energy Phys.* **04** (2002) 060 [hep-th/0202151].

[16] E. Fuchs, M. Kroyter and A. Marcus, *Squeezed state projectors in string field theory*, *J. High Energy Phys.* **09** (2002) 022 [hep-th/0207001].

[17] B. Chen and F.-L. Lin, *D-branes as GMS solitons in vacuum string field theory*, *Phys. Rev.* **D 66** (2002) 126001 [hep-th/0204233].

[18] L. Bonora, D. Mamone and M. Salizzoni, *Vacuum string field theory ancestors of the GMS solitons*, *J. High Energy Phys.* **01** (2003) 013 [hep-th/0207044].

[19] I. Bars, *Map of Witten's $*$ to Moyal's $*$*, *Phys. Lett.* **B 517** (2001) 436 [hep-th/0106157].

[20] I. Bars and Y. Matsuo, *Computing in string field theory using the Moyal star product*, *Phys. Rev.* **D 66** (2002) 066003 [hep-th/0204260].

[21] M.R. Douglas, H. Liu, G. Moore and B. Zwiebach, *Open string star as a continuous Moyal product*, *J. High Energy Phys.* **04** (2002) 022 [hep-th/0202087].

[22] J. Klusoň, *Some remarks about Berkovits' superstring field theory*, *J. High Energy Phys.* **06** (2001) 045 [hep-th/0105319].

[23] M. Mariño and R. Schiappa, *Towards vacuum superstring field theory: the supersliver*, *J. Math. Phys.* **44** (2003) 156 [hep-th/0112231].

JHEP07(2003)061

[24] I.Ya. Aref'eva, D.M. Belov and A.A. Giryavets, *Construction of the vacuum string field theory on a non-BPS brane*, *J. High Energy Phys.* **09** (2002) 050 [hep-th/0201197].

[25] N. Berkovits, *Super-Poincaré invariant superstring field theory*, *Nucl. Phys.* **B 450** (1995) 90 [hep-th/9503099].

[26] E. Witten, *Interacting field theory of open superstrings*, *Nucl. Phys.* **B 276** (1986) 291.

[27] I.Ya. Aref'eva, P.B. Medvedev and A.P. Zubarev, *New representation for string field solves the consistence problem for open superstring field*, *Nucl. Phys.* **B 341** (1990) 464.

[28] N. Berkovits, A. Sen and B. Zwiebach, *Tachyon condensation in superstring field theory*, *Nucl. Phys.* **B 587** (2000) 147 [hep-th/0002211].

[29] P.-J. De Smet and J. Raeymaekers, *Level four approximation to the tachyon potential in superstring field theory*, *J. High Energy Phys.* **05** (2000) 051 [hep-th/0003220].

[30] I.Ya. Aref'eva, A.S. Koshelev, D.M. Belov and P.B. Medvedev, *Tachyon condensation in cubic superstring field theory*, *Nucl. Phys.* **B 638** (2002) 3 [hep-th/0011117].

[31] J. Klusoň, *Some solutions of Berkovits' superstring field theory*, hep-th/0201054.

[32] I.Ya. Aref'eva, A.A. Giryavets and A.S. Koshelev, *NS ghost slivers*, *Phys. Lett.* **B 536** (2002) 138 [hep-th/0203227].

[33] K. Ohmori, *Comments on solutions of vacuum superstring field theory*, *J. High Energy Phys.* **04** (2002) 059 [hep-th/0204138].

[34] K. Ohmori, *On ghost structure of vacuum superstring field theory*, *Nucl. Phys.* **B 648** (2003) 94 [hep-th/0208009].

[35] K. Ohmori, *Level-expansion analysis in NS superstring field theory revisited*, hep-th/0305103.

[36] N. Berkovits and W. Siegel, *Covariant field theory for self-dual strings*, *Nucl. Phys.* **B 505** (1997) 139 [hep-th/9703154].

[37] D.J. Gross and A. Jevicki, *Operator formulation of interacting string field theory*, *Nucl. Phys.* **B 283** (1987) 1.

[38] D.M. Belov, *Diagonal representation of open string star and Moyal product*, hep-th/0204164.

[39] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Ghost structure and closed strings in vacuum string field theory*, hep-th/0111129.

[40] N. Berkovits and C. Vafa, $N = 4$ *topological strings*, *Nucl. Phys.* **B 433** (1995) 123 [hep-th/9407190].

[41] O. Lechtenfeld, A.D. Popov and S. Uhlmann, *Exact solutions of Berkovits' string field theory*, *Nucl. Phys.* **B 637** (2002) 119 [hep-th/0204155].

[42] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[43] D.J. Gross and A. Jevicki, *Operator formulation of interacting string field theory, 3. NSR superstring*, *Nucl. Phys.* **B 293** (1987) 29.

[44] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories, 1. Foundations*, *Nucl. Phys.* **B 505** (1997) 569 [hep-th/9705038].

JHEP07(2003)061

[45] I. Kishimoto, *Some properties of string field algebra*, *J. High Energy Phys.* **12** (2001) 007 [hep-th/0110124].

[46] L. Bonora, C. Maccaferri, D. Mamone and M. Salizzoni, *Topics in string field theory*, hep-th/0304270.

[47] V.A. Kostelecký and R. Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, *Phys. Rev.* **D 63** (2001) 046007 [hep-th/0008252].

[48] S. Mandelstam, *Interacting string picture of dual resonance models*, *Nucl. Phys.* **B 64** (1973) 205.

[49] S. Mandelstam, *Lorentz properties of the three-string vertex*, *Nucl. Phys.* **B 83** (1974) 413.

[50] S. Mandelstam, *Dual-resonance models*, *Phys. Rept.* **13** (1974) 259.

[51] M. Kaku and K. Kikkawa, *The field theory of relativistic strings. Pt. 2: loops and pomerons*, *Phys. Rev.* **D 10** (1974) 1823.

[52] E. Cremmer and J.-L. Gervais, *Combining and splitting relativistic strings*, *Nucl. Phys.* **B 76** (1974) 209.

[53] E. Cremmer and J.-L. Gervais, *Infinite component field theory of interacting relativistic strings and dual theory*, *Nucl. Phys.* **B 90** (1975) 410.

[54] A. LeClair, M.E. Peskin and C.R. Preitschopf, *String field theory on the conformal plane, 1. Kinematical principles*, *Nucl. Phys.* **B 317** (1989) 411.

[55] A. LeClair, M.E. Peskin and C.R. Preitschopf, *String field theory on the conformal plane, 2. Generalized gluing*, *Nucl. Phys.* **B 317** (1989) 464.

[56] K. Okuyama, *Ghost kinetic operator of vacuum string field theory*, *J. High Energy Phys.* **01** (2002) 027 [hep-th/0201015].

[57] J. Bordes, A. Abdurrahman and F. Anton, *N string vertices in string field theory*, *Phys. Rev.* **D 49** (1994) 2966 [hep-th/9306029].

[58] M. Ihl, A. Kling and S. Uhlmann, to appear.

[59] O. Lechtenfeld and A.D. Popov, *On the integrability of covariant field theory for open $N = 2$ strings*, *Phys. Lett.* **B 494** (2000) 148 [hep-th/0009144].

[60] A. Kling, O. Lechtenfeld, A.D. Popov and S. Uhlmann, *On nonperturbative solutions of superstring field theory*, *Phys. Lett.* **B 551** (2003) 193 [hep-th/0209186].

[61] A. Kling, O. Lechtenfeld, A.D. Popov and S. Uhlmann, *Solving string field equations: new uses for old tools*, *Fortschr. Phys.* **51** (2003) 775 [hep-th/0212335].

[62] D.J. Gross and A. Jevicki, *Operator formulation of interacting string field theory, 2*, *Nucl. Phys.* **B 287** (1987) 225.

[63] S. Samuel, *The ghost vertex in E. Witten's string field theory*, *Phys. Lett.* **B 181** (1986) 255.

[64] C. Maccaferri and D. Mamone, *Star democracy in open string field theory*, hep-th/0306252.

JHEP07(2003)061